

# THE QUASICONFORMAL INVARIANT PROPERTIES OF JOHN DOMAINS IN $\mathbb{R}^n$

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**ABSTRACT.** Suppose that  $D$  is a proper domain in  $\mathbb{R}^n$  and that  $f$  is a quasiconformal mapping from  $D$  onto a John domain  $D'$  in  $\mathbb{R}^n$ . The main aim of this paper is to prove that if  $D_1 \subset D$  is a John domain, then the image  $f(D_1)$  of  $D_1$  under  $f$  is still a John domain. This result shows that the answer to one of the open problems raised by Heinonen in [10] is affirmative.

## 1. INTRODUCTION AND MAIN RESULTS

Throughout the paper, we always assume that  $D$  is a proper subdomain in  $\mathbb{R}^n$  and  $\mathbb{B}(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$  denotes the open ball centered at  $x_0$  with radius  $r > 0$ . Similarly, for the closed balls and spheres, we use the notations  $\bar{\mathbb{B}}(x_0, r)$  and  $\partial\mathbb{B}(x_0, r)$ , respectively. In particular, we use  $\mathbb{B}$  to denote the unit ball  $\mathbb{B}(0, 1)$ . We begin with the following concepts.

**Definition 1.1.** A domain  $D$  in  $\mathbb{R}^n$  is said to be *c-uniform* if there exists a constant  $c$  with the property that each pair of points  $z_1, z_2$  in  $D$  can be joined by a rectifiable arc  $\gamma$  in  $D$  satisfying (cf. [19, 25])

- (1)  $\min_{j=1,2} \ell(\gamma[z_j, z]) \leq c d_D(z)$  for all  $z \in \gamma$ , and
- (2)  $\ell(\gamma) \leq c |z_1 - z_2|$ ,

where  $\ell(\gamma)$  denotes the arclength of  $\gamma$ ,  $\gamma[z_j, z]$  the part of  $\gamma$  between  $z_j$  and  $z$ , and  $d_D(z)$  the distance from  $z$  to the boundary  $\partial D$  of  $D$ . Also we say that  $\gamma$  is a *double c-cone arc*.

A domain  $D$  in  $\mathbb{R}^n$  is said to be a *c-John domain* if it satisfies the condition (1) in Definition 1.1, but not necessarily (2), and  $\gamma$  is called a *c-cone arc*.

John [13], and Martio and Sarvas [19] were the first who introduced and studied John domains and uniform domains, respectively. Now, there are plenty of alternative characterizations for uniform and John domains, see [3, 5, 7, 16, 18, 25, 26, 27, 28, 29]. And its importance along with some special domains throughout the function theory is well documented, see [5, 16, 20, 22]. Moreover, John domains and

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uniform domains in  $\mathbb{R}^n$  enjoy with numerous geometric and function theoretic features in many areas of modern mathematical analysis, see [2, 3, 7, 9, 14, 15, 22, 28] (see also [1]).

In the following, we always assume that  $f : D \rightarrow D' \subset \mathbb{R}^n$  is a  $K$ -quasiconformal mapping with  $K \geq 1$ . See [21, 31] for definitions and properties of  $K$ -quasiconformal mappings. In 1985, Väisälä proved

**Theorem A.** ([22, Theorem 5.6]) *If  $D'$  is uniform, then for each uniform subdomain  $D_1$  in  $D$ ,  $f(D_1)$  is still uniform.*

In [4], the authors showed the following

**Theorem B.** ([4, pp. 120-121]) *Suppose  $D'$  is QED. Then for each QED subdomain  $D_1$  in  $D$ ,  $f(D_1)$  is also QED.*

The case (1) of the following result is due to Väisälä [24, Theorem 2.20] whereas the case (2) is obtained by Heinonen [10, Theorem 7.1].

**Theorem C.** *Suppose  $D'$  is broad.*

- (1) *Then for each John subdomain  $D_1$  in  $D$ ,  $f(D_1)$  is a John domain;*
- (2) *If both  $D$  and  $D'$  are bounded, then for each broad subdomain  $D_1$  in  $D$ ,  $f(D_1)$  is a broad domain.*

We refer to [19] for an early discussions on this topic. However, a natural problem is that whether  $f(D_1)$  is a John domain for each John subdomain  $D_1$  of  $D$  when  $D'$  is John. In fact, this is an open problem raised by Heinonen [10] in the following form:

**Open Problem 1.1.** Suppose that  $f$  is a quasiconformal mapping of a domain  $D$  in  $\mathbb{R}^n$  onto a John domain  $D'$  in  $\mathbb{R}^n$ . Is it then true that every John subdomain of  $D$  is mapped onto a John subdomain of  $D'$  by  $f$ ?

Heinonen himself discussed this problem and as consequence he obtained the following

**Theorem D.** ([10, Theorem 7.3]) *Let  $f : \mathbb{B} \rightarrow D'$  be a quasiconformal mapping onto a John domain  $D'$ , and let  $C_M(w)$  denote the Stolz cone with vertex at  $w \in \partial\mathbb{B}$ . Then  $f(C_M(w))$  is uniform.*

Here the *Stolz cone*  $C_M(w)$  with vertex at  $w \in \partial\mathbb{B}$  is defined to be the interior of the closed convex hull of  $w$  and the hyperbolic ball centered at 0 with radius  $M > 0$ .

The main aim of this paper is to show that the answer to Open Problem 1.1 is yes.

**Theorem 1.1.** *Suppose  $D$  is a proper subdomain in  $\mathbb{R}^n$ . If  $f : D \rightarrow D'$  is a quasiconformal mapping onto a John domain  $D'$ , then the image of each John subdomain of  $D$  under  $f$  is still a John subdomain in  $D'$ .*

We present the proof of Theorem 1.1 in Section 3. In Section 2, we present some useful results and prove several necessary lemmas. As a continuation of the present

article, it is possible to obtain several interesting applications of Theorem 1.1. As an indication to this, we state the following result whose proof will be presented in a later article.

**Theorem 1.2.** *Suppose  $D'$  is an inner uniform domain and  $D_1$  is an inner uniform subdomain in  $D$ . Then each quasihyperbolic geodesic in  $D'_1$  is inner quasiconvex.*

The following is a consequence of the last two theorems.

**Corollary 1.1.** *Suppose  $D'$  is an inner uniform domain and  $D_1$  is an inner uniform subdomain in  $D$ . Then  $D'_1$  is still inner uniform.*

## 2. PRELIMINARIES

Definition 1.1 is often referred to as the “arclength” definition for uniform domains and John domains. When the word “arclength” in Definition 1.1 is replaced by “diameter”, then it is called the “diameter” definition for uniform domains and John domains.

The following result reveals the close relationship between these two definitions.

**Theorem E.** ([19, 23]) *The “arclength” definition for uniform domains and John domains is quantitatively equivalent to the “diameter” one.*

Let  $\gamma$  be a rectifiable arc or path in  $D$ . Then the *quasihyperbolic length* of  $\gamma$  is defined to be the number  $\ell_{k_D}(\gamma)$  given by (cf. [8]):

$$\ell_{k_D}(\gamma) = \int_{\gamma} \frac{|dz|}{d_D(z)}.$$

For  $z_1, z_2$  in  $D$ , the *quasihyperbolic distance*  $k_D(z_1, z_2)$  between  $z_1$  and  $z_2$  is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_{k_D}(\gamma),$$

where the infimum is taken over all rectifiable arcs  $\gamma$  joining  $z_1$  to  $z_2$  in  $D$ . An arc  $\gamma$  from  $z_1$  to  $z_2$  is called a *quasihyperbolic geodesic* if  $\ell_{k_D}(\gamma) = k_D(z_1, z_2)$ . Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between two points in  $D$  always exists (cf. [7, Lemma 1]). Moreover, for every quasigeodesic  $\gamma$  in  $D$  joining  $z_1$  to  $z_2$ , we have

$$(2.1) \quad k_D(z_1, z_2) \geq \log \left( 1 + \frac{\ell(\gamma)}{\min\{d_D(z_1), d_D(z_2)\}} \right).$$

In particular, for  $z_1, z_2$  in  $D$ , we have (cf. [25, 31])

$$k_D(z_1, z_2) \geq \log \left( 1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right) \geq \left| \log \frac{d_D(z_2)}{d_D(z_1)} \right|.$$

As a generalization of quasiconformal mappings, Väisälä introduced CQH homeomorphisms (cf. [25]).

**Definition 2.1.** Suppose  $f : D \rightarrow D'$  is a homeomorphism. Then  $f$  is said to be  $C$ -coarsely  $M$ -quasihyperbolic, or briefly  $(M, C)$ -CQH, in the quasihyperbolic metric if it satisfies

$$\frac{k_D(x, y) - C}{M} \leq k_{D'}(f(x), f(y)) \leq M k_D(x, y) + C$$

for all  $x, y \in D$ .

The following proposition easily follows from [7, Theorem 3].

**Proposition 2.1.** *Each  $K$ -quasiconformal mapping in  $\mathbb{R}^n$  is an  $(M, C)$ -CQH homeomorphism with  $(M, C)$  depending only on  $(K, n)$ .*

**Theorem F.** ([30, 2.50 (2)]) *A domain  $D \subset \mathbb{R}^n$  is  $c$ -uniform if and only if there is a constant  $\mu_1(c)$  such that for all  $x, y \in D$ ,*

$$k_D(x, y) \leq \mu_1(c) \log \left( 1 + \frac{|x - y|}{\min\{d_D(x), d_D(y)\}} \right),$$

where  $\mu_1(c)$  is a constant depending on  $c$ .

This form of the definition of the uniform domain is due to Gehring and Osgood [7]. As a matter of fact, in [7, Theorem 1], there was an additive constant in the inequality of Theorem F, but it was shown by Vuorinen in [30, 2.50 (2)] that the additive constant can be chosen to be zero.

**Theorem G.** ([7, Theorem 3]) *Suppose that  $G$  and  $G'$  are domains in  $\mathbb{R}^n$ , and that  $f : G \rightarrow G'$  is a  $K$ -quasiconformal mapping. Then for all  $z_1, z_2 \in G$ ,*

$$k_{G'}(z'_1, z'_2) \leq \mu_2 \max\{k_G(z_1, z_2), (k_G(z_1, z_2))^{\frac{1}{\mu_2}}\},$$

where the constant  $\mu_2 \geq 1$  depends on  $K$  and  $n$ .

Suppose that  $G$  denotes a domain in  $\mathbb{R}^n$ ,  $E$  and  $F$  are two disjoint continua in  $G$  and  $\text{Mod}(E, F; G)$  denotes the usual conformal modulus of the family of all curves joining  $E$  and  $F$  in  $G$ . The following two related results are useful for us.

**Theorem H.** ([11, p. 397], see also §11.9 in [21], §7 in [31] and [30, Lemmas 2.39 (1) and 2.44]) *For each  $n \geq 2$ , there exist decreasing homeomorphisms  $\phi_n, \psi_n : (0, \infty) \rightarrow (0, \infty)$  such that*

$$\phi_n(t) \leq \text{Mod}(E, F; \mathbb{R}^n) \leq \psi_n(t),$$

where “ $\text{dist}$ ” (resp. “ $\text{diam}$ ”) means “distance” (resp. “diameter”) and

$$t = \frac{\text{dist}(E, F)}{\min\{\text{diam}E, \text{diam}F\}}.$$

**Theorem I.** ([6, Theorem 4.15], and [11, p. 397]) *Suppose that  $G \subset \mathbb{R}^n$  is a  $c$ -uniform domain. Then*

$$\text{Mod}(E, F; \mathbb{R}^n) \leq \mu_3(n, c) \text{Mod}(E, F; G)$$

for every pair of continua  $E$  and  $F$  in  $G$ , where  $\mu_3(n, c)$  is a constant depending on  $n$  and  $c$ .

**Definition 2.2.** Let  $X_1$  and  $X_2$  be two metric spaces with distance written as  $|x - y|$  for  $x \in X_1$  and  $y \in X_2$ , and let  $\eta : [0, \infty) \rightarrow [0, \infty)$  be a homeomorphism. An embedding  $f : X_1 \rightarrow X_2$  is  $\eta$ -quasisymmetric if

$$|a - x| \leq t|a - y| \text{ implies } |f(a) - f(x)| \leq \eta(t)|f(a) - f(y)| \text{ for all } a, x, y \in X_1.$$

If there is a constant  $\nu \geq 1$  such that

$$|a - x| \leq |a - y| \text{ implies } |f(a) - f(x)| \leq \nu|f(a) - f(y)|,$$

then  $f$  is said to be *weakly  $\nu$ -quasisymmetric*.

**Theorem J.** ([22, Theorem 5.6]) *Suppose that  $n \geq 2$ ,  $G$  is a  $c_1$ -uniform domain in  $\mathbb{R}^n$  and that  $f$  is a  $K$ -quasiconformal homeomorphism of  $G$  onto a domain  $G' \subset \mathbb{R}^n$ . If both  $G$  and  $G'$  are bounded and  $G'$  is  $c_2$ -uniform, then  $f$  is  $\eta$ -quasisymmetric, where the homeomorphism  $\eta = \eta_{K, c_1, c_2}$  depends on  $K$ ,  $c_1$  and  $c_2$ .*

For  $x, y$  in  $D$ , the internal metric  $\delta_D$  in  $D$  is defined by

$$\delta_D(x, y) = \inf\{\text{diam}(\alpha) : \alpha \subset D \text{ is a rectifiable arc joining } x \text{ and } y\}.$$

**Definition 2.3.** Let  $\varphi : (0, \infty) \rightarrow (0, \infty)$  be a decreasing homeomorphism. We say that  $D$  is  $\varphi$ -broad if for each  $t > 0$  and each pair  $(C_0, C_1)$  of continua in  $D$  the condition  $\delta_D(C_0, C_1) \leq t \min\{\text{diam}(C_0), \text{diam}(C_1)\}$  implies

$$\text{Mod}(C_0, C_1; D) \geq \varphi(t),$$

where  $\delta_D(C_0, C_1)$  denotes the  $\delta_D$ -distance between  $C_0$  and  $C_1$ .

Here is a result concerning  $\varphi$ -broad due to Gehring and Martio [6].

**Theorem K.** ([6, Lemma 2.6]) *If  $D$  is a  $c$ -uniform domain, then  $D$  is  $\varphi(c)$ -broad, where  $\varphi(c)$  depends on  $c$ .*

**Definition 2.4.** Suppose that  $A \subset D$  and  $b \geq 1$  is a constant. We say that  $A$  is  $b$ -LLC<sub>2</sub> (resp.  $b$ -LLC<sub>2</sub> with respect to  $\delta_D$ ) in  $D$  if for all  $x \in A$  and  $r > 0$ , the points in  $A \setminus \overline{\mathbb{B}}(x, br)$  (resp.  $A \setminus \overline{\mathbb{B}}_{\delta_D}(x, br)$ ) can be joined in  $D \setminus \overline{\mathbb{B}}(x, r)$  (resp.  $D \setminus \overline{\mathbb{B}}_{\delta_D}(x, r)$ ). If  $A = D$ , then we say that  $D$  is  $b$ -LLC<sub>2</sub> (resp.  $b$ -LLC<sub>2</sub> with respect to  $\delta_D$ ), where

$$\mathbb{B}_{\delta_D}(x, r) = \{z \in \mathbb{R}^n : \delta_D(z, x) < r\}.$$

The next three theorems are crucial for further discussions in our investigation.

**Theorem L.** ([10, Theorem 6.1]) *Suppose that  $D$  and  $D'$  are bounded, that  $f : D \rightarrow D'$  is  $K$ -quasiconformal, and that  $D$  is  $\varphi$ -broad. If  $A \subset D$  is such that  $f(A)$  is  $b$ -LLC<sub>2</sub> with respect to  $\delta_{D'}$  in  $D'$ , then  $f|_A : A \rightarrow f(A)$  is weakly  $H$ -quasisymmetric in the metrics  $\delta_D$  and  $\delta_{D'}$  with  $H$  depending only on the data*

$$\mu_4 = \mu_4 \left( n, K, b, \varphi, \frac{\delta_D(A)}{d_D(x_0)}, \frac{\delta_{D'}(f(A))}{d_{D'}(f(x_0))} \right),$$

where  $x_0$  is a fixed point in  $A$  and  $\delta_D(A)$  denotes the  $\delta_D$ -diameter of  $A$ .

**Theorem M.** ([10, Theorem 6.2]) *Let  $D \subset \mathbb{R}^n$  and let  $A \subset D$  be such that each  $x \in A$  can be joined with a fixed point  $x_0$  in  $D$  by a  $b$ -carrot in  $D$ . Then  $A$  is both  $\mu_5(b)$ -LLC<sub>2</sub> and  $\mu_5(b)$ -LLC<sub>2</sub> with respect to  $\delta_D$  in  $D$ , where  $\mu_5(b)$  depends on  $b$ .*

**Theorem N.** ([25, Theorem 4.15]) *For proper subdomains  $D$  and  $D'$  in Banach spaces, suppose that  $f : D \rightarrow D'$  is  $(M, C)$ -CQH. If  $\gamma$  is a  $(c_0, h)$ -solid arc in  $D$ , then the image arc  $\gamma'$  of  $\gamma$  under  $f$  is  $(c'_0, h_1)$ -solid in  $D'$  with  $(c'_0, h_1)$  depending only on  $(c_0, h, M, C)$ .*

For convenience, in the following, we always assume that  $x, y, z, \dots$  denote points in a domain  $D$  in  $\mathbb{R}^n$  and  $x', y', z', \dots$  the images in  $D'$  of  $x, y, z, \dots$  under  $f$ , respectively. Also we assume that  $\alpha, \beta, \gamma, \dots$  denote curves in  $D$  and  $\alpha', \beta', \gamma', \dots$  the images in  $D'$  of  $\alpha, \beta, \gamma, \dots$  under  $f$ , respectively.

Let  $G$  be a domain in  $\mathbb{R}^n$ . For  $x, y \in G$ , let  $\beta$  be a quasihyperbolic geodesic joining  $x$  and  $y$  in  $G$ . Suppose that  $G'$  is a  $c$ -uniform domain and  $f : G \rightarrow G'$  is a  $K$ -quasiconformal mapping. Without loss of generality, we may assume that  $d_{G'}(y') \geq d_{G'}(x')$ . Let  $x'_0 \in \beta'$  be the first point along the direction from  $x'$  to  $y'$  such that

$$d_{G'}(x'_0) = \sup_{p' \in \beta'} d_{G'}(p').$$

It is possible that  $x'_0 = x'$  or  $y'$ . Clearly, there exists a nonnegative integer  $m$  such that

$$2^m d(x') \leq d(x'_0) < 2^{m+1} d(x'),$$

and  $w'_0$  the first point in  $\beta'[x', x'_0]$  from  $x'$  to  $x'_0$  with

$$d(w'_0) = 2^m d(x').$$

Let  $x'_1 = x'$ . If  $w'_0 = x'_1$ , we let  $x'_2 = x'_0$ . It is possible that  $x'_1 = x'_2$ . If  $w'_0 \neq x'_1$ , then we let  $x'_2, \dots, x'_{m+1} \in \beta'[x', x'_0]$  be the points such that for each  $i \in \{2, \dots, m+1\}$ ,  $x'_i$  denotes the first point from  $x'$  to  $x'_0$  with

$$d(x'_i) = 2^{i-1} d(x'_1).$$

Obviously,  $x'_{m+1} = w'_0$ . If  $w'_0 \neq x'_0$ , then we use  $x'_{m+2}$  to denote  $x'_0$ .

In a similar way, let  $s \geq 0$  be the integer such that

$$2^s d(y') \leq d(x'_0) < 2^{s+1} d(y'),$$

and  $y'_{1,0}$  the first point in  $\beta'[y', x'_0]$  from  $y'$  to  $x'_0$  with

$$d(y'_{1,0}) = 2^s d(y').$$

Let  $y'_{1,1} = y'$ . If  $y'_{1,0} = y'_{1,1}$ , we let  $y'_{1,2} = y'_{1,0}$ . It is possible that  $y'_{1,2} = y'_{1,1}$ . If  $y'_{1,0} \neq y'$ , then we let  $y'_{1,2}, \dots, y'_{1,s+1}$  be the points in  $\beta'[y', x'_0]$  such that for each  $j \in \{2, \dots, s+1\}$ ,  $y'_{1,j}$  is the first point from  $y'_{1,1}$  to  $x'_0$  with

$$d(y'_{1,j}) = 2^{j-1} d(y'_{1,1}).$$

Then  $y'_{1,s+1} = y'_{1,0}$ . If  $y'_{1,0} \neq x'_0$ , we let  $y'_{1,s+2} = x'_0$ .

**2.1. Elementary properties.** By Proposition 2.1, in the following, we may assume that  $f : G \rightarrow G'$  is an  $(M, C)$ -CQH homeomorphism, where  $(M, C)$  depends only on  $(K, n)$ .

**Lemma 2.1.** *For any  $k \in \{1, \dots, m\}$  and  $z' \in \beta'[x'_k, x'_{k+1}]$ ,*

- (1)  $d_{G'}(x'_{k+1}) \leq \left(1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1}\right)^{\mu_1(c)M^2} e^{CM+C} d_{G'}(z')$ ;
- (2)  $|x'_{k+1} - x'_k| \leq \left(1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1}\right)^{\mu_1(c)M^2+1} e^{CM+C} d_{G'}(z')$ ; and
- (3)  $\max\{|x'_k - z'|, |x'_{k+1} - z'|\} \leq \left(e^{h_1} + (1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1})^{\mu_1(c)M^2+1} \times e^{CM+C}\right) d_{G'}(z')$ .

*Proof.* At first, we prove the following inequality by the method of contradiction:

$$(2.2) \quad |x'_{k+1} - x'_k| < (1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1} d_{G'}(x'_{k+1}).$$

Suppose on the contrary that

$$(2.3) \quad |x'_{k+1} - x'_k| \geq (1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1} d_{G'}(x'_{k+1}).$$

Let  $\{z'_{k,i}\}$ , where  $i \in \{1, \dots, [\mu_1(c)c'_0 + 4]^2 + 2\}$ , denote  $[(\mu_1(c)c'_0 + 4)^2] + 2$  successive points in  $\beta'[x'_k, x'_{k+1}]$  with  $z'_{k,1} = x'_k$ ,  $z'_{k,[(\mu_1(c)c'_0 + 4)^2] + 2} = x'_{k+1}$  and for any  $i \in \{1, \dots, [(\mu_1(c)c'_0 + 4)^2] + 1\}$ ,  $|z'_{k,i+1} - z'_{k,i}| \geq \frac{|x'_k - x'_{k+1}|}{[(\mu_1(c)c'_0 + 4)^2] + 1}$ . In what follows,  $[\cdot]$  always denotes the greatest integer part. Then for each  $i \in \{1, 2, \dots, [(\mu_1(c)c'_0 + 4)^2] + 1\}$ ,

$$\begin{aligned} k_{G'}(z'_{k,i}, z'_{k,i+1}) &\geq \log \left( 1 + \frac{|z'_{k,i+1} - z'_{k,i}|}{\min\{d_{G'}(z'_{k,i+1}), d_{G'}(z'_{k,i})\}} \right) \\ &\geq \log \left( 1 + \frac{|x'_k - x'_{k+1}|}{2([(\mu_1(c)c'_0 + 4)^2] + 1)d_{G'}(x'_k)} \right) \\ &> h_1, \end{aligned}$$

from which we infer that

$$\sum_{i=1}^{[(\mu_1(c)c'_0 + 4)^2] + 1} k_{G'}(z'_{k,i}, z'_{k,i+1}) \leq c'_0 k_{G'}(x'_k, x'_{k+1}).$$

Hence Theorem F shows that

$$\begin{aligned} (2.4) \quad &([(\mu_1(c)c'_0 + 4)^2] + 1) \log \left( 1 + \frac{|x'_{k+1} - x'_k|}{2([(\mu_1(c)c'_0 + 4)^2] + 1)d_{G'}(x'_k)} \right) \\ &\leq \mu_1(c)c'_0 \log \left( 1 + \frac{|x'_{k+1} - x'_k|}{d_{G'}(x'_k)} \right). \end{aligned}$$

Elementary computations and (2.3) yield

$$\begin{aligned}
& ([(\mu_1(c)c'_0 + 4)^2] + 1) \log \left( 1 + \frac{|x'_{k+1} - x'_k|}{2([(\mu_1(c)c'_0 + 4)^2] + 1)d_{G'}(x'_k)} \right) \\
& > \mu_1(c)c'_0 \log \left( 1 + \frac{|x'_{k+1} - x'_k|}{d_{G'}(x'_k)} \right)
\end{aligned}$$

which contradicts (2.4). Hence (2.2) holds.

Since  $\beta$  is a quasihyperbolic geodesic and  $f$  is an  $(M, C)$ -CQH homeomorphism, it follows from (2.2) and Theorem F that for any  $z' \in \beta'[x'_k, x'_{k+1}]$ ,

$$\begin{aligned}
\log \frac{d_{G'}(x'_{k+1})}{d_{G'}(z')} &< k_{G'}(z', x'_{k+1}) \\
&\leq Mk_G(z, x_{k+1}) + C \\
&\leq M k_G(x_k, x_{k+1}) + C \\
&\leq M^2 k_{G'}(x'_k, x'_{k+1}) + CM + C \\
&\leq \mu_1(c)M^2 \log \left( 1 + \frac{|x'_{k+1} - x'_k|}{d_{G'}(x'_k)} \right) + CM + C \\
&\leq \mu_1(c)M^2 \log (1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1}) + MC + C,
\end{aligned}$$

which implies that Lemma 2.1(1) holds.

It follows from (2.2) and Lemma 2.1(1) that

$$|x'_{k+1} - x'_k| \leq \left( 1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1} \right)^{\mu_1(c)M^2+1} e^{CM+C} d_{G'}(z'),$$

from which Lemma 2.1(2) follows.

To prove Lemma 2.1(3), without loss of generality, we may assume that

$$\min\{|x'_k - z'|, |x'_{k+1} - z'|\} = |x'_{k+1} - z'|.$$

If  $\log \left( 1 + \frac{|x'_{k+1} - z'|}{d_{G'}(z')} \right) \leq h_1$ , then  $|x'_{k+1} - z'| \leq (e^{h_1} - 1)d_{G'}(z')$ . By Lemma 2.1(2), we have

$$\begin{aligned}
(2.5) \quad |x'_k - z'| &\leq |x'_{k+1} - z'| + |x'_{k+1} - x'_k| \\
&\leq \left( e^{h_1} + (1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1})^{\mu_1(c)M^2+1} e^{MC+C} \right) d_{G'}(z').
\end{aligned}$$

If  $\log \left( 1 + \frac{|x'_{k+1} - z'|}{d_{G'}(z')} \right) > h_1$ , then  $\log \left( 1 + \frac{|x'_k - z'|}{d_{G'}(z')} \right) > h_1$  and so

$$\begin{aligned}
\log \left( 1 + \frac{|x'_k - z'|}{d_{G'}(z')} \right) &\leq k_{G'}(x'_k, z') \\
&\leq c'_0 k_{G'}(x'_k, x'_{k+1}) \\
&\leq \mu_1(c)c'_0 \log \left( 1 + \frac{|x'_{k+1} - x'_k|}{d_{G'}(x'_k)} \right)
\end{aligned}$$

which, together with (2.2), yields the inequality:

$$(2.6) \quad |x'_k - z'| \leq \left( 1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1} \right)^{\mu_1(c)c'_0} d_{G'}(z').$$



Inequalities (2.5) and (2.6) show that Lemma 2.1(3) holds.  $\square$

In the following, we let

$$\mu = \left( e^{h_1} + (1 + 2(1 + (2\mu_1(c)c'_0 + 8)^3)e^{h_1})^{\mu_1(c)M^2+1} e^{CM+C} \right).$$

The following two results easily follow from the similar reasoning as in the proof of Lemma 2.1.

**Corollary 2.1.** *For any  $k \in \{1, \dots, s\}$  and  $z' \in \beta'[y'_{1,k}, y'_{1,k+1}]$ , we have*

- (1)  $d_{G'}(y'_{1,k+1}) \leq \mu d_{G'}(z')$ ;
- (2)  $|y'_{1,k+1} - y'_{1,k}| \leq \mu d_{G'}(z')$ ; and
- (3)  $\max\{|y'_{1,k} - z'|, |y'_{1,k+1} - z'|\} \leq \mu d_{G'}(z')$ .

**Corollary 2.2.** *For any  $z' \in \beta'[x'_{m+1}, y'_{1,s+1}]$ , the following hold.*

- (1)  $d_{G'}(x'_0) \leq \mu d_{G'}(z')$ ;
- (2)  $|x'_{m+1} - y'_{1,s+1}| \leq \mu d_{G'}(z')$ ; and
- (3)  $\max\{|x'_{m+1} - z'|, |y'_{1,s+1} - z'|\} \leq \mu d_{G'}(z')$ .

**Lemma 2.2.** *For any  $z' \in \beta'[x', x'_0]$ ,*

$$|x' - z'| \leq \mu_6 d_{G'}(z'),$$

*and for any  $z' \in \beta'[y', x'_0]$ ,*

$$|y' - z'| \leq \mu_6 d_{G'}(z'),$$

*where  $\mu_6 = \mu + \mu^2$ .*

*Proof.* We only need to prove the first assertion since the proof for the second one is similar.

If  $z' \in \beta'[x', x'_{m+1}]$ , then there exists some  $k \in \{1, \dots, m\}$  such that  $z' \in \beta'[x'_k, x'_{k+1}]$ . If  $k = 1$ , then the result easily follows from Lemma 2.1. If  $k > 1$ , then, by Lemma 2.1,

$$\begin{aligned} |x' - z'| &\leq |x'_1 - x'_2| + \dots + |x'_{k-1} - x'_k| + |x'_k - z'| \\ &\leq \mu(d_{G'}(x'_1) + \dots + d_{G'}(x'_{k-1}) + d_{G'}(z')) \\ &\leq (\mu + \mu^2)d_{G'}(z'). \end{aligned}$$

Now we consider the case  $z' \in \beta'[x'_{m+1}, x'_0]$ . Then we infer from Lemma 2.1 and Corollary 2.2 that

$$\begin{aligned} |x' - z'| &\leq \mu(d_{G'}(x'_1) + d_{G'}(x'_2) + \dots + d_{G'}(x'_m) + d_{G'}(z')) \\ &\leq \mu(d_{G'}(x'_{m+1}) + d_{G'}(z')) \\ &\leq (\mu + \mu^2)d_{G'}(z'). \end{aligned}$$

Hence the first assertion in Lemma 2.2 holds.  $\square$

**Lemma 2.3.** *Suppose that  $|x' - y'| \geq \frac{1}{2}d_{G'}(y')$ . Then for every  $z' \in \beta'$ ,*

$$|y' - z'| + |z' - x'| \leq \rho_1 |x' - y'|,$$

*where  $\rho_1 = 6(1 + 3\mu_6)^{\mu_1(c)\mu_2^2} + 2$ .*

*Proof.* We need to deal with two cases separately.

**Case 2.1.**  $\min\{|x' - x'_0|, |x'_0 - y'|\} \leq |x' - y'|$ .

It follows from the hypothesis “ $d_{G'}(y') \geq d_{G'}(x')$ ” that

$$d_{G'}(x'_0) \leq \min\{|x' - x'_0|, |x'_0 - y'|\} + d_{G'}(y') \leq 3|x' - y'|,$$

whence Lemma 2.2 implies

$$(2.7) \quad \begin{aligned} |y' - z'| + |x' - z'| &\leq 2\mu_6 d_{G'}(z') + |x' - y'| \\ &\leq (6\mu_6 + 1)|x' - y'|. \end{aligned}$$

**Case 2.2.**  $\min\{|x' - x'_0|, |x'_0 - y'|\} > |x' - y'|$ .

Let  $x'_{0,1}$  (resp.  $x'_{0,2}$ ) be the first point in  $\beta'[x', x'_0]$  from  $x'$  to  $x'_0$  (resp.  $\beta'[y', x'_0]$  from  $y'$  to  $x'_0$ ) such that  $|x'_{0,1} - x'| = |x' - y'|$  (resp.  $|x'_{0,2} - y'| = |x' - y'|$ ). Then it follows from Theorems F and Lemma 2.2 that

$$\begin{aligned} k_{G'}(x'_{0,1}, x'_{0,2}) &\leq \mu_1(c) \log \left( 1 + \frac{|x'_{0,1} - x'_{0,2}|}{\min\{d_{G'}(x'_{0,1}), d_{G'}(x'_{0,2})\}} \right) \\ &\leq \mu_1(c) \log \left( 1 + \mu_6 \frac{|x'_{0,1} - x'_{0,2}|}{|x' - y'|} \right) \\ &\leq \mu_1(c) \log (1 + 3\mu_6). \end{aligned}$$

For all  $w' \in \beta'[x'_{0,1}, x'_{0,2}]$ , it follows from Theorem G that

$$k_G(w, x_{0,1}) \leq k_G(x_{0,1}, x_{0,2}) \leq \mu_1(c)\mu_2 \log(1 + 3\mu_6),$$

whence

$$\log \left( 1 + \frac{|w' - x'_{0,1}|}{d_{G'}(x'_{0,1})} \right) \leq k_{G'}(w', x'_{0,1}) \leq \mu_1(c)\mu_2^2 \log (1 + 3\mu_6)$$

which implies

$$(2.8) \quad \begin{aligned} |w' - x'_{0,1}| &\leq (1 + 3\mu_6)^{\mu_1(c)\mu_2^2} d_{G'}(x'_{0,1}) \\ &\leq 3(1 + 3\mu_6)^{\mu_1(c)\mu_2^2} |x' - y'|, \end{aligned}$$

since  $d_{G'}(x'_{0,1}) \leq d_{G'}(x') + |x'_{0,1} - x'| \leq 3|x' - y'|$ .

Similarly, we have

$$(2.9) \quad \begin{aligned} |w' - x'_{0,2}| &\leq (1 + 3\mu_6)^{\mu_1(c)\mu_2^2} d_{G'}(x'_{0,2}) \\ &\leq 3(1 + 3\mu_6)^{\mu_1(c)\mu_2^2} |x' - y'|. \end{aligned}$$

If  $z' \in \beta'[x', x'_{0,1}]$ , then

$$(2.10) \quad |y' - z'| + |x' - z'| \leq |x' - y'| + 2|x' - z'| \leq 3|x' - y'|.$$

Similarly, if  $z' \in \beta'[y', x'_{0,2}]$ , then

$$(2.11) \quad |y' - z'| + |x' - z'| \leq 3|x' - y'|.$$

Finally, the case we need to consider is when  $z' \in \beta'[x'_{0,1}, x'_{0,2}]$ . In this case, the inequalities (2.8) and (2.9) show that

$$\begin{aligned}
 (2.12) \quad |y' - z'| + |x' - z'| &\leq |z' - x'_{0,1}| + |x'_{0,1} - x'| + |z' - x'_{0,2}| \\
 &\quad + |x'_{0,2} - y'| \\
 &\leq 2 \left( 3(1 + 3\mu_6)^{\mu_1(c)\mu_2^2} + 1 \right) |x' - y'|.
 \end{aligned}$$

The combination of (2.7), (2.10), (2.11) and (2.12) completes the proof of Lemma 2.3.  $\square$

**Lemma 2.4.** *Suppose that for  $x_1, x_2 \in G$ , the following hold:*

- (1)  $\overline{\mathbb{B}}(x_1, r_1) \cap \overline{\mathbb{B}}(x_2, r_2) \neq \emptyset$ ;
- (2)  $\frac{1}{3\rho_2}d_G(x_1) \leq r_1 \leq \frac{1}{\rho_2}d_G(x_1)$ ;
- (3)  $\frac{1}{3\rho_2}d_G(x_2) \leq r_2 \leq \frac{1}{\rho_2}d_G(x_2)$ ,

where  $\rho_2 = (2^8 c)^2$ . Then

$$\frac{7}{8}d_G(x_2) \leq d_G(x_1) \leq \frac{8}{7}d_G(x_2) \quad \text{and} \quad \frac{1}{4}r_1 \leq r_2 \leq 4r_1.$$

*Proof.* For each  $y \in \partial\mathbb{B}(x_1, r_1) \cap \overline{\mathbb{B}}(x_2, r_2)$ , since

$$d_G(y) \geq d_G(x_2) - r_2, \quad d_G(x_1) \geq d_G(y) - r_1$$

and

$$d_G(y) \geq d_G(x_1) - r_1, \quad d_G(x_2) \geq d_G(y) - r_2,$$

we see that the lemma holds.  $\square$

**Lemma 2.5.** *Suppose that for  $x_1, x_2 \in G$ , the following hold:*

- (1)  $\frac{1}{3\rho_2}d_G(x_1) \leq r_1 \leq \frac{1}{\rho_2}d_G(x_1)$ ,
- (2)  $\frac{1}{3\rho_2}d_G(x_2) \leq r_2 \leq \frac{1}{\rho_2}d_G(x_2)$ , and
- (3)  $\text{dist}(\mathbb{B}(x_1, r_1), \mathbb{B}(x_2, r_2)) \geq \frac{1}{\mu_7}r_1$  for  $\mu_7 \geq 1$ .

Then, we have

$$\text{dist}(\mathbb{B}(x_1, r_1), \mathbb{B}(x_2, r_2)) \geq \frac{1}{4\mu_7}r_2.$$

*Proof.* Denote the intersection point of  $[x_1, x_2]$  with the sphere  $\partial\mathbb{B}(x_1, r_1)$  by  $z$ , where  $[x_1, x_2]$  means the segment with the endpoints  $x_1$  and  $x_2$ . From the fact

$$d_G(x_2) - |z - x_2| \leq d_G(y) \leq d_G(x_1) + |x_1 - z|,$$

the proof easily follows.  $\square$

For each  $\mu_8$ -cone arc  $\eta \subset G$  with endpoints  $x_1$  and  $x_2$  in  $G$ , where  $\mu_8 (\geq 1)$  is a constant, let  $s_0$  bisect  $\eta$ . Then, we obtain

**Lemma 2.6.** (1) *For every  $u \in \eta[x_1, s_0]$ ,*

$$d_G(u) \geq \frac{2\ell(\eta[x_1, u]) + d_G(x_1)}{4\mu_8};$$

(2) *For every  $u \in \eta[s_0, x_2]$ ,*

$$d_G(u) \geq \frac{2\ell(\eta[x_2, u]) + d_G(x_2)}{4\mu_8}.$$

*Proof.* It suffices to prove the first statement since the proof for the second one is similar. For every  $u \in \eta[x_1, s_0]$ ,

$$d_G(u) \geq \frac{\ell(\eta[x_1, u])}{\mu_8}.$$

If  $\eta[x_1, u] \subset \mathbb{B}(u, \frac{1}{2}d_G(x_1))$ , then  $d_G(u) \geq (1/2)d_G(x_1)$ . Otherwise,

$$d_G(u) \geq \frac{1}{2\mu_8}d_G(x_1).$$

Hence

$$d_G(u) \geq \frac{2\ell(\eta[x_1, u]) + d_G(x_1)}{4\mu_8}$$

and so Lemma 2.6 holds.  $\square$

In the rest of this paper, we always assume that  $D_1 \subset D$  is a  $c$ -John domain. For  $z_1, z_2 \in D_1$ , let  $\alpha \subset D_1$  denote a rectifiable arc with endpoints  $z_1$  and  $z_2$  satisfying

$$\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c d_{D_1}(z) \quad \text{for all } z \in \alpha,$$

i.e.  $\alpha$  is a  $c$ -cone arc, and let  $z_0$  bisect  $\alpha$ . Denote  $\alpha[z_1, z_0]$  and  $\alpha[z_2, z_0]$  by  $\gamma$  and  $\beta$ , respectively.

The following two lemmas play an active role in the proof of our main result, Theorem 1.1. Moreover these lemmas are the improved forms of the corresponding ones in [12].

**Lemma 2.7.** *There exists a simply connected domain  $D_{1,0} = \bigcup_{i=1}^{k_1} B_{1,i} \subset D_1$  such that*

- (1)  $z_1, z_0 \in \overline{D_{1,0}}$ , the closure of  $D_{1,0}$ ;
- (2) For each  $i \in \{1, \dots, k_1\}$ ,

$$\frac{1}{3\rho_2} d_{D_1}(x_{1,i}) \leq r_{1,i} \leq \frac{1}{\rho_2} d_{D_1}(x_{1,i});$$

- (3) If  $k_1 \geq 3$ , then for all  $i, j \in \{1, \dots, k_1\}$  with  $|i - j| \geq 2$ ,

$$\text{dist}(B_{1,i}, B_{1,j}) \geq \frac{1}{2^6 \rho_3} \max\{r_{1,i}, r_{1,j}\};$$

(4) If  $k_1 \geq 2$ , then

$$r_{1,i} + r_{1,i+1} - |x_{1,i} - x_{1,i+1}| \geq \frac{1}{2^6 \rho_3} \max\{r_{1,i}, r_{1,i+1}\}$$

for each  $i \in \{1, \dots, k_1 - 1\}$ ,

where  $B_{1,i} = \mathbb{B}(x_{1,i}, r_{1,i})$ ,  $x_{1,i} \in \gamma$ ,  $x_{1,i} \notin B_{1,i-1}$  for each  $i \in \{2, \dots, k_1\}$  and  $\rho_3 = 10[\rho_2 c]$ .

*Proof.* Let  $x_{1,1} = z_1$ . Set  $A_{1,1} = \mathbb{B}(x_{1,1}, r_{1,1})$  with  $r_{1,1} = \frac{1}{2\rho_2} d_{D_1}(x_{1,1})$ .

If  $z_0 \in \overline{A_{1,1}}$ , then we let  $B_{1,1} = A_{1,1}$ , and the domain  $D_{1,0} = B_{1,1}$  is the desired.

If  $z_0 \notin \overline{A_{1,1}}$ , then we let  $x_{1,2}$  be the last intersection point of  $\gamma$  from  $z_1$  to  $z_0$  with  $\partial A_{1,1}$ . Set  $A_{1,2} = \mathbb{B}(x_{1,2}, r_{1,2})$  with  $r_{1,2} = \frac{1}{2\rho_2} d_{D_1}(x_{1,2})$ .

If  $z_0 \in \overline{A_{1,2}}$  and  $A_{1,1}$  is contained in  $A_{1,2}$ , then we let  $B_{1,1} = A_{1,2}$ , and the domain  $D_{1,0} = B_{1,1}$  is the needed. If  $z_0 \in \overline{A_{1,2}}$  and  $A_{1,1}$  is not contained in  $A_{1,2}$ , then we let  $B_{1,1} = A_{1,1}$ ,  $B_{1,2} = A_{1,2}$ , and the domain  $D_{1,0} = B_{1,1} \cup B_{1,2}$  is our desired.

If  $z_0 \notin \overline{A_{1,2}}$ , then we let  $x_{1,3}$  be the last intersection point of  $\gamma$  from  $z_1$  to  $z_0$  with  $\partial A_{1,2}$ . Set  $A_{1,3} = \mathbb{B}(x_{1,3}, r_{1,3})$  with  $r_{1,3} = \frac{1}{2\rho_2} d_{D_1}(x_{1,3})$ ;  $\dots$ .

We continue this procedure until there is some  $i \in \{1, \dots, s-2\}$  such that

$$\text{dist}(A_{1,i}, A_{1,s}) < \frac{1}{2^6 \rho_3} \max\{r_{1,i}, r_{1,s}\}.$$

Obviously,  $s \geq 3$ . We have the following claim.

**Claim 2.1.** *There are  $q$  balls  $C_{1,1}, \dots, C_{1,q}$  among  $A_{1,1}, \dots, A_{1,s}$  such that the conditions (2), (3) and (4) in the lemma are satisfied (possibly,  $q = 1$ ).*

In order to prove this claim, we let  $A_{1,t}$  be the first ball from  $A_{1,1}$  to  $A_{1,s-2}$  such that  $\overline{A_{1,t}} \cap \overline{A_{1,s}} \neq \emptyset$ . If  $A_{1,t} \subset A_{1,s}$ , then  $t = 1$  and we let  $C_{1,1} = A_{1,s}$ . So  $q = 1$  in this case. Otherwise, we divide the discussions into two cases.

**Case 2.3.**  $r_{1,t} + r_{1,s} - |x_{1,t} - x_{1,s}| \geq \frac{1}{8} r_{1,s}$ .

We let  $C_{1,i} = A_{1,i}$  for each  $i \in \{1, \dots, t\}$  and  $C_{1,t+1} = \mathbb{B}(x_{1,s}, \frac{15}{16} r_{1,s})$ . Then Lemmas 2.4 and 2.5 show that the balls  $C_{1,1}, C_{1,2}, \dots, C_{1,t}$  and  $C_{1,t+1}$  satisfy the conditions (2), (3) and (4) in Lemma 2.7, and so  $q = t + 1$ .

**Case 2.4.**  $r_{1,t} + r_{1,s} - |x_{1,t} - x_{1,s}| < \frac{1}{8} r_{1,s}$ .

In this case, we consider the ball  $A_{1,s}^* = \mathbb{B}(x_{1,s}, 2r_{1,s})$ . Let  $A_{1,s_1}$  be the first ball from  $A_{1,1}$  to  $A_{1,t}$ , whose closure  $\overline{A_{1,s_1}}$  has nonempty intersection with  $\overline{A_{1,s}^*}$ . For each  $i \in \{s_1, \dots, t\}$ , we denote  $\text{dist}(A_{1,i}, A_{1,s})$  by  $d_i$ . Clearly,  $d_i = 0$  if and only if  $i = t$ . We divide the rest of the discussions into two subcases.

**Subcase 2.1.**  $d_{s_1} \leq \frac{3}{16} r_{1,s}$ .

In this case, we take  $C_{1,i} = A_{1,i}$  ( $1 \leq i \leq s_1$ ) and  $C_{1,s_1+1} = \mathbb{B}(x_{1,s}, \frac{4}{3} r_{1,s})$ . Then the balls  $C_{1,1}, C_{1,2}, \dots, C_{1,s_1}, C_{1,s_1+1}$  satisfy the conditions (2), (3) and (4) in our lemma. So  $q = s_1 + 1$ .

**Subcase 2.2.**  $d_{s_1} > \frac{3}{16}r_{1,s}$ .

Let  $\delta_1 = d_{s_1}$  and  $\delta_2$  be the first  $d_i$  from  $d_{s_1}$  to  $d_t$  satisfying  $d_i < \delta_1$ . Clearly,  $\delta_1 > \delta_2$ . By repeating the procedure, we get  $\delta_1, \dots, \delta_m \in \{d_{s_1}, \dots, d_t\}$  such that

$$\delta_1 > \delta_2 > \dots > \delta_m = 0.$$

For each  $h \in \{1, \dots, m-1\}$ , we denote  $A_{1,i_h} = \mathbb{B}(x_{1,i_h}, r_{1,i_h})$  the first ball from  $A_{1,1}$  to  $A_{1,t}$  with  $d_{i_h} = \delta_h$ , and define  $\varepsilon_h = \delta_h - \delta_{h+1}$ .

**Subclaim 2.1.** *There must exist some  $j \in \{1, \dots, m-1\}$  such that  $\varepsilon_j > \frac{1}{8\rho_3}r_{1,s}$ .*

If  $m \leq \rho_3$ , then the existence of  $j \in \{1, \dots, m-1\}$  with  $\varepsilon_j > \frac{1}{8\rho_3}r_{1,s}$  is obvious because otherwise,

$$\frac{3}{16}r_{1,s} < \delta_1 - \delta_m \leq \frac{m-1}{8\rho_3}r_{1,s} < \frac{1}{8}r_{1,s}$$

which is a contradiction.

For the remaining case, namely,  $m > \rho_3$ , we suppose on the contrary that  $\varepsilon_h \leq \frac{1}{8\rho_3}r_{1,s}$  for all  $h \in \{1, \dots, m-1\}$ . Note that

$$\delta_{m-\rho_3} - \delta_m = \varepsilon_{m-\rho_3} + \dots + \varepsilon_{m-1} \leq \frac{1}{8}r_{1,s}.$$

Then for all  $h \in \{m-\rho_3, \dots, m-1\}$ ,

$$\delta_h \leq \frac{1}{8}r_{1,s}.$$

If there exists some  $h \in \{m-\rho_3, \dots, m-1\}$  such that

$$A_{1,i_h} = \mathbb{B}(x_{1,i_h}, r_{1,i_h}) \not\subset \{A_{1,s}^* \setminus A_{1,s}\},$$

then  $(A_{1,s}^* \setminus A_{1,s}) \cap A_{1,i_h}$  contains a ball, denoted by

$$\mathbb{B}(x_{0,i_h}, r_{0,i_h}) \text{ with } r_{0,i_h} = \frac{r_{1,s} - \delta_h}{2} > \frac{7}{16}r_{1,s}.$$

Hence  $r_{1,i_h} \geq \frac{7}{16}r_{1,s}$ . On the other hand, if  $A_{1,i_h} = \mathbb{B}(x_{1,i_h}, r_{1,i_h}) \subset \{A_{1,s}^* \setminus A_{1,s}\}$  for some  $h \in \{m-\rho_3, \dots, m-1\}$  then we see that  $r_{1,i_h} > \frac{1}{4}r_{1,s}$ , because otherwise,

$$\frac{1}{4}r_{1,s} \geq r_{1,i_h} \geq \frac{1}{3\rho_2}d_{D_1}(x_{1,i_h}) \geq \frac{1}{3\rho_2}\left(1 - \frac{2}{\rho_2}\right)d_{D_1}(x_{1,s}) > \frac{1}{4}r_{1,s}$$

which is clearly a contradiction. We have thus proved that for each  $h \in \{m-\rho_3, \dots, m-1\}$ ,

$$r_{1,i_h} > \frac{1}{4}r_{1,s}.$$

Hence

$$\begin{aligned} 2c\rho_2r_{1,s} &= cd_{D_1}(x_{1,s}) \\ &\geq \ell(\gamma[z_1, x_{1,s}]) \\ &\geq \frac{\rho_3 - 1}{4}r_{1,s}, \end{aligned}$$

which is the desired contradiction. The proof of Subclaim 2.1 is complete.

We come back to the proof of Claim 2.1. Let  $j$  be the least number in  $\{1, \dots, m-1\}$  satisfying Subclaim 2.1 and let  $A_{1,s}^\dagger = \mathbb{B}(x_{1,s}, r_{1,s}^\dagger)$ , where

$$r_{1,s}^\dagger = r_{1,s} + \delta_{j+1} + \frac{1}{16\rho_3} r_{1,s}.$$

By Subclaim 2.1, we see that for all  $i < i_{j+1}$ ,

$$\text{dist}(A_{1,s}^\dagger, B_{1,i}) \geq \frac{1}{16\rho_3} r_{1,s}.$$

We choose the following balls: Let  $C_{1,i} = A_{1,i}$  for each  $i \in \{1, \dots, i_{j+1}\}$  and  $C_{1,i_{j+1}+1} = A_{1,s}^\dagger$ . It follows from Lemmas 2.4 and 2.5 that the balls  $C_{1,1}, \dots, C_{1,i_{j+1}}, C_{1,i_{j+1}+1}$  satisfy the conditions (2), (3) and (4) in the lemma. So  $q = i_{j+1} + 1$  in this case.

The proof of Claim 2.1 is finished.

We continue the proof of our lemma.

We consider the balls  $C_{1,1}, \dots, C_{1,q}$  which are obtained in Claim 2.1 from  $A_{1,1}, \dots, A_{1,s}$ .

If  $z_0 \in \overline{C}_{1,q}$ , then by letting  $B_{1,i} = C_{1,i}$  for each  $i \in \{1, \dots, q\}$ , we see that the domain  $D_{1,0} = \bigcup_{i=1}^q B_{1,i}$  is the desired.

If  $z_0 \notin \overline{C}_{1,q}$ , then we let  $x_{1,q+1}$  be the last intersection point of  $\gamma$  from  $z_1$  to  $z_0$  with  $\partial C_{1,q}$ . Set  $C_{1,q+1} = \mathbb{B}(x_{1,q+1}, r_{1,q+1})$  with  $r_{1,q+1} = \frac{1}{2\rho_2} d_{D_1}(x_{1,q+1})$ ;  $\dots$

By repeating the procedure as above, we will get a set of points  $\{x_{1,i}\}_{i=1}^{k_1}$  on  $\gamma$  and a set of balls  $\{C_{1,i} = \mathbb{B}(x_{1,i}, r_{1,i})\}_{i=1}^{k_1}$  in  $D$  such that Conditions (2), (3) and (4) are satisfied and  $z_0$  is contained in the closure of  $C_{1,k_1}$ . By letting  $B_{1,i} = C_{1,i}$  for each  $i \in \{1, \dots, k_1\}$ , we know that  $D_{1,0} = \bigcup_{i=1}^{k_1} B_{1,i}$  is the needed domain. Hence Lemma 2.7 holds.  $\square$

A similar argument as in the proof of Lemma 2.7 gives

**Lemma 2.8.** *There exists a simply connected domain  $D_{2,0} = \bigcup_{u=1}^{k_2} B_{2,u} \subset D_1$  such that*

- (1)  $z_2, z_0 \in \overline{D}_{2,0}$ , the closure of  $D_{2,0}$ ;
- (2) For each  $u \in \{1, \dots, k_2\}$ ,

$$\frac{1}{3\rho_2} d_{D_1}(x_{2,u}) \leq r_{2,u} \leq \frac{1}{\rho_2} d_{D_1}(x_{2,u});$$

- (3) If  $k_2 \geq 3$ , then for all  $u, v \in \{1, \dots, k_2\}$  with  $|u - v| \geq 2$ ,

$$\text{dist}(B_{2,u}, B_{2,v}) \geq \frac{1}{2^6 \rho_3} \max\{r_{1,u}, r_{1,v}\};$$

- (4) If  $k_2 \geq 2$ , then for each  $u \in \{1, \dots, k_2 - 1\}$ ,

$$r_{2,u} + r_{2,u+1} - |x_{2,u} - x_{2,u+1}| \geq \frac{1}{2^6 \rho_3} \max\{r_{2,u}, r_{2,u+1}\},$$

where  $B_{2,u} = \mathbb{B}(x_{2,u}, r_{2,u})$ ,  $x_{2,u} \in \beta$  and  $x_{2,u} \notin B_{2,u-1}$ .

In order to prove the next lemma we need the following result which is from [17].

**Theorem O.** ([17, Theorem 1.2]) *Suppose that  $E_1$  and  $E_2$  are two convex domains in a Banach space, where  $E_1$  is bounded and  $E_2$  is  $c$ -uniform for some  $c > 1$ , and suppose that there exist  $z_0 \in E_1 \cap E_2$  and  $r > 0$  such that  $\mathbb{B}(z_0, r) \subset E_1 \cap E_2$ . If there exist constants  $R_1 > 0$  and  $c_0 > 1$  such that  $R_1 \leq c_0 r$  and  $E_1 \subset \mathbb{B}(z_0, R_1)$ , then  $E_1 \cup E_2$  is a  $\rho_4$ -uniform domain with  $\rho_4 = (c + 1)(2c_0 + 1) + c$ .*

**Lemma 2.9.**  $D_{1,0}$  is a  $\rho_5$ -uniform domain, where  $\rho_5 = 2^{10}c\rho_2\rho_3$ .

*Proof.* It suffices to prove that for all  $u_1$  and  $u_2 \in D_{1,0}$ , there is a double  $\rho_5$ -cone arc  $\varphi$  in  $D_{1,0}$  connecting  $u_1$  and  $u_2$ .

We first consider the case where there is an  $i \in \{1, \dots, k_1 - 1\}$  such that  $u_1, u_2 \in B_{1,i} \cup B_{1,i+1}$ . Under this assumption, clearly, the existence of  $\varphi$  follows from Theorem O.

Next, we consider the case where there are  $i, j \in \{1, \dots, k\}$  such that  $j - i \geq 2$ ,  $u_1 \in B_{1,i}$ ,  $u_2 \in B_{1,j}$  and  $\{u_1, u_2\}$  is not contained in  $B_{1,t} \cup B_{1,t+1}$  for all  $t \in \{i, \dots, j - 1\}$ . It suffices to prove the case:  $u_1 \notin [x_{1,i}, x_{1,i+1}]$  and  $u_2 \notin [x_{1,j-1}, x_{1,j}]$  since the discussions for other cases are similar.

Let

$$\varphi = [u_1, x_{1,i}] \cup [x_{1,i}, x_{1,i+1}] \cup \dots \cup [x_{1,j-1}, x_{1,j}] \cup [x_{1,j}, u_2].$$

For each  $u \in \varphi$ , if  $u \in [u_1, x_{1,i}]$ , then

$$\ell(\varphi[u_1, u]) = |u_1 - u| < d_{D_{1,0}}(u).$$

Also if  $u \in [u_2, x_{1,j}]$ , then

$$\ell(\varphi[u_2, u]) = |u_2 - u| < d_{D_{1,0}}(u).$$

In the following, we consider the case  $u \in \varphi[x_{1,i}, x_{1,j}]$ . Clearly, there exists a  $t \in \{i, \dots, j\}$  such that  $u \in B_{1,t}$ . If  $u \in B_{1,j}$ , then by Lemma 2.7,

$$\ell(\varphi[u_2, u]) \leq 2r_{1,j} \leq 2^8\rho_3d_{D_{1,0}}(u).$$

Similarly, if  $u \in B_{1,i}$ , then

$$\ell(\varphi[u_1, u]) \leq 2r_{1,i} \leq 2^8\rho_3d_{D_{1,0}}(u).$$

So it remains to consider the case:  $u \in \varphi \setminus (B_{1,i} \cup B_{1,j})$ . Then Lemma 2.7 implies

$$\begin{aligned} \ell(\varphi[u_1, u]) &\leq \ell(\varphi[u_1, x_{1,t}]) + r_{1,t} \\ &\leq 2cd_{D_1}(x_{1,t}) + r_{1,t} \\ &\leq (6c\rho_2 + 1)r_{1,t} \\ &\leq 2^7(6c\rho_2 + 1)\rho_3d_{D_{1,0}}(u). \end{aligned}$$

Hence  $\varphi$  is a  $2^7(6c\rho_2 + 1)\rho_3$ -cone arc. It follows from Lemma 2.7 that

$$\max\{r_{1,i}, r_{1,j}\} \leq 2^6\rho_3|u_1 - u_2|,$$



whence

$$\begin{aligned}
\ell(\varphi[u_1, u_2]) &\leq r_{1,i} + \ell(\varphi[x_{1,i}, x_{1,j}]) + r_{1,j} \\
&\leq 2^7 \rho_3 |u_1 - u_2| + cd_{D_1}(x_{1,j}) \\
&\leq 2^7 \rho_3 |u_1 - u_2| + 3c\rho_2 r_{1,j} \\
&\leq 2^8 c\rho_2 \rho_3 |u_1 - u_2|.
\end{aligned}$$

The proof of the existence of  $\varphi$  is finished.  $\square$

The following corollary is a consequence of Lemmas 2.2, 2.9 and Theorem F.

**Corollary 2.3.** (1) For all  $x$  and  $y \in D_{1,0}$ ,

$$k_{D_{1,0}}(x, y) \leq \mu_9 \log \left( 1 + \frac{|x - y|}{\min\{d_{D_{1,0}}(x), d_{D_{1,0}}(y)\}} \right);$$

(2) For  $w_1, w_2 \in D_{1,0}$ , suppose that  $\eta' \subset D'_{1,0}$  is a quasihyperbolic geodesic joining  $w'_1$  and  $w'_2$  in  $D'_{1,0}$  and that  $v_{0,0} \in \eta$  satisfies

$$d_{D_{1,0}}(v_{0,0}) = \sup_{p \in \eta} d_{D_{1,0}}(p).$$

Then, for  $j = 1, 2$ , we have

$$|w_j - z| \leq \mu_9 d_{D_{1,0}}(z) \quad \text{for all } z \in \eta[w_j, v_{0,0}],$$

where  $\mu_9$  is a constant depending on  $\rho_5, K$  and  $n$ .

The following result easily follows from a similar argument as in the proof of Lemma 2.9. So, we omit its proof.

**Lemma 2.10.** The domain  $D_{2,0}$  defined in Lemma 2.8 is  $\rho_5$ -uniform, where  $\rho_5$  is the same as in Lemma 2.9.

For all  $i \in \{2, \dots, k_1 - 1\}$ , let  $x_{1,i-1}, x_{1,i}$  and  $x_{1,i+1}$  denote the centers of the balls  $B_{1,i-1}, B_{1,i}$  and  $B_{1,i+1}$ , respectively, as constructed in Lemma 2.7. If  $x_{1,i-1}, x_{1,i}$  and  $x_{1,i+1}$  are not collinear, we use  $\theta_i := \angle x_{1,i-1}x_{1,i}x_{1,i+1}$  to denote the angle determined by  $x_{1,i-1}, x_{1,i}$  and  $x_{1,i+1}$  with  $\theta_i < \pi$ . We let  $\theta_i = \pi$  if they are collinear. Then we obtain

**Lemma 2.11.** If  $\theta_i \leq \pi/2$ , then  $\sin \theta_i \geq 1/(2^7 \rho_3)$  for  $i \in \{2, \dots, k_1 - 1\}$ .

*Proof.* Let  $T$  be the 2-dimensional subspace containing  $x_{1,i-1}, x_{1,i}$  and  $x_{1,i+1}$ . Then we let  $x_0, x_1 \in \mathbb{S}(x_{1,i+1}, r_{1,i+1}) \cap \mathbb{S}(x_{1,i}, r_{1,i}) \cap T$ . The hypothesis implies that  $[x_0, x_1]$  is orthogonal to  $[x_{1,i}, x_{1,i+1}]$ . Let  $x_2$  be the intersection point of  $[x_{1,i}, x_{1,i+1}]$  with  $[x_0, x_1]$ . It follows from Lemma 2.7 that

$$|x_2 - x_0| \geq \frac{1}{2}(r_{1,i} + r_{1,i+1} - |x_{1,i} - x_{1,i+1}|) \geq \frac{1}{2^7 \rho_3} \max\{r_{1,i}, r_{1,i+1}\}.$$

Hence

$$\sin \theta_i \geq \sin \angle x_0 x_{1,i} x_{1,i+1} = \sin \angle x_1 x_{1,i} x_{1,i+1} \geq \frac{1}{2^7 \rho_3}$$

and the proof is complete.  $\square$

## 3. PROOF OF THEOREM 1.1

In what follows, we assume that  $D_1 \subset D$  is a  $c$ -John domain and  $D'$  is an  $a$ -John domain. For  $z_1$  and  $z_2 \in D_1$ , we will construct a cone arc in  $D'_1$  to join  $z'_1$  and  $z'_2$  in the sense of “diameter”. Clearly, this approach shows that  $D'_1$  is a John domain. In order to construct such a cone arc, we let  $\alpha \subset D_1$  be a  $c$ -cone arc with endpoints  $z_1$  and  $z_2$ ,  $z_0$  bisect  $\alpha$ , and let  $\gamma \triangleq \alpha[z_1, z_0]$ ,  $\beta \triangleq \alpha[z_2, z_0]$ . We assume further that  $D_{1,0}$  and  $D_{2,0}$  are the simply connected domains constructed as in Lemmas 2.7 and 2.8, respectively. Set

$$\gamma_{1,0} = [z_1, x_{1,2}] \cup \cdots \cup [x_{1,k_1-1}, x_{1,k_1}] \cup [x_{1,k_1}, z_0]$$

and

$$\beta_{1,0} = [z_2, x_{2,2}] \cup \cdots \cup [x_{2,k_2-1}, x_{2,k_2}] \cup [x_{2,k_2}, z_0].$$

Obviously,  $\gamma_{1,0} \setminus \{z_0\} \subset D_{1,0}$  and  $\beta_{1,0} \setminus \{z_0\} \subset D_{2,0}$ . Finally, we will construct the needed cone arc based on  $\gamma'_{1,0}$  and  $\beta'_{1,0}$ . In order to complete the task, we need to prove several lemmas.

**Lemma 3.1.** *For each  $z \in D_{1,0}$ , we have*

$$d_{D_1}(z) \geq (\rho_2 - 1)d_{D_{1,0}}(z),$$

and for each  $z \in \gamma_{1,0}[z_1, x_{1,k_1}]$ ,

- (1)  $d_{D_1}(z) \leq 2^7(3\rho_2 + 1)\rho_3 d_{D_{1,0}}(z)$ ;
- (2)  $d_{D'_1}(z') \leq (2^8(3\rho_2 + 1)\rho_3\mu_2)^{\mu_2} d_{D'_{1,0}}(z')$ .

*Proof.* For every  $z \in D_{1,0}$ , by Lemma 2.7, there exists a  $j \in \{1, \dots, k_1\}$  such that  $z \in B_{1,j}$ . Without loss of generality, we may assume that  $B_{1,i}$  is the last ball from  $B_{1,1}$  to  $B_{1,k_1}$  such that  $z \in B_{1,i}$ . Then

$$d_{D_1}(z) \geq d_{D_1}(x_{1,i}) - |z - x_{1,i}| \geq (\rho_2 - 1)d_{D_{1,0}}(x_{1,i}) \geq (\rho_2 - 1)d_{D_{1,0}}(z).$$

For all  $z \in \gamma_{1,0}[z_1, x_{1,k_1}]$ , we still assume that  $B_{1,i}$  is the last ball from  $B_{1,1}$  to  $B_{1,k_1}$  such that  $z \in B_{1,i}$ . Again, Lemma 2.7 implies

$$d_{D_{1,0}}(z) \geq \frac{1}{2^7\rho_3}r_{1,i}$$

and

$$d_{D_1}(z) \leq d_{D_1}(x_{1,i}) + r_{1,i} \leq (3\rho_2 + 1)r_{1,i},$$

whence

$$d_{D_1}(z) \leq 2^7(3\rho_2 + 1)\rho_3 d_{D_{1,0}}(z)$$

which shows that (1) holds.

Since for all  $u \in \mathbb{S}(z, d_{D_{1,0}}(z))$

$$\begin{aligned} \log \left( 1 + \frac{1}{2^7(3\rho_2 + 1)\rho_3} \right) &\leq \log \left( 1 + \frac{|u - z|}{d_{D_1}(z)} \right) \\ &\leq k_{D_1}(z, u) \\ &\leq \int_{[z, u]} \frac{|dw|}{d_{D_1}(w)} \\ &\leq \frac{1}{\rho_2 - 2}, \end{aligned}$$

we see from Theorem G that

$$k_{D'_1}(z', u') \geq \left( \frac{1}{\mu_2} \log \left( 1 + \frac{1}{2^7(3\rho_2 + 1)\rho_3} \right) \right)^{\mu_2} > \frac{1}{((2^8(3\rho_2 + 1)\rho_3)\mu_2)^{\mu_2} - 1},$$

whence

$$u' \in \mathbb{R}^n \setminus \overline{\mathbb{B}} \left( z', \frac{1}{(2^8(3\rho_2 + 1)\rho_3\mu_2)^{\mu_2}} d_{D'_1}(z') \right).$$

Then the proof of (2) easily follows from the fact that  $f(\overline{\mathbb{B}}(z, d_{D_{1,0}}(z))) \subset D'_{1,0}$ .  $\square$

**Lemma 3.2.** *For each  $z \in \gamma_{1,0}$ ,  $\ell(\gamma_{1,0}[z_1, z]) \leq \rho_6 d_{D_1}(z)$ , where  $\rho_6 = \frac{8}{7}c$ .*

*Proof.* Clearly, for every  $z \in \gamma_{1,0}$ , there exists an  $i \in \{1, \dots, k_1\}$  such that  $z \in \mathbb{B}(x_{1,i}, r_{1,i})$ . It follows from Lemma 2.7 that

$$\ell(\gamma_{1,0}[z_1, z]) \leq \ell(\gamma[z_1, x_{1,i}]) + r_{1,i} \leq \left( c + \frac{1}{\rho_2} \right) d_{D_1}(x_{1,i}) \leq \rho_6 d_{D_1}(z),$$

from which the proof follows.  $\square$

The following result easily follows from Lemma 3.2 and the similar reasoning as in the proof of Lemma 2.6.

**Lemma 3.3.** *If  $u \in \gamma_{1,0}[x_1, x_2]$  for  $x_1$  and  $x_2 \in \gamma_{1,0}$ , then*

$$\begin{aligned} (1) \quad d_{D_1}(u) &\geq \frac{2\ell(\gamma_{1,0}[x_1, u]) + d_{D_1}(x_1)}{4\rho_6}, \text{ and} \\ (2) \quad d_D(u) &\geq \frac{2\ell(\gamma_{1,0}[x_1, u]) + d_D(x_1)}{4\rho_6}. \end{aligned}$$

**Lemma 3.4.** *For all  $u_1, u_2 \in \gamma_{1,0}$ ,  $\gamma_{1,0}[u_1, u_2]$  is a double  $2^8 c \rho_2 \rho_3$ -cone arc in  $D_1$ .*

*Proof.* Clearly, for all  $u_1, u_2 \in \gamma_{1,0}$ , Lemma 3.2 implies that for each  $z \in \gamma_{1,0}[u_1, u_2]$ ,

$$(3.1) \quad \ell(\gamma_{1,0}[u_1, z]) \leq \ell(\gamma_{1,0}[z_1, z]) \leq \rho_6 d_{D_1}(z).$$

We need to prove

$$(3.2) \quad \ell(\gamma_{1,0}[u_1, u_2]) \leq 2^8 c \rho_2 \rho_3 |u_1 - u_2|.$$

We first consider the case where there is an  $i \in \{1, \dots, k_1 - 1\}$  such that  $u_1, u_2 \in B_{1,i} \cup B_{1,i+1}$ . Without loss of generality, we may assume that  $i > 1$  since the proof for the case  $i = 1$  is similar.

If  $u_1$  and  $u_2$  lie on the segment  $[x_{1,i-1}, x_{1,i}]$  or  $[x_{1,i}, x_{1,i+1}]$ , then

$$(3.3) \quad \ell(\gamma_{1,0}[u_1, u_2]) = |u_1 - u_2|.$$

Now we assume that  $\{u_1, u_2\}$  is contained neither in  $[x_{1,i-1}, x_{1,i}]$  nor in  $[x_{1,i}, x_{1,i+1}]$ .

If  $\theta_i = \angle x_{1,i-1}x_{1,i}x_{1,i+1} \geq \frac{\pi}{2}$ , then

$$(3.4) \quad \ell(\gamma_{1,0}[u_1, u_2]) \leq 2|u_1 - u_2|.$$

If  $\theta_i < \frac{\pi}{2}$ , then Lemma 2.11 implies that

$$(3.5) \quad \begin{aligned} |u_1 - u_2| &\geq \max\{|u_2 - x_{1,i}|, |u_1 - x_{1,i}|\} \sin \theta_i \\ &\geq \frac{1}{2^8 \rho_3} \ell(\gamma_{1,0}[u_1, u_2]). \end{aligned}$$

The remaining is to consider the case where there are  $i, j \in \{1, \dots, k_1\}$  such that  $j - i \geq 2$ ,  $u_1 \in B_{1,i}$ ,  $u_2 \in B_{1,j}$  and  $\{u_1, u_2\}$  is not contained in  $B_{1,t} \cup B_{1,t+1}$  for all  $t \in \{i, \dots, j-1\}$ . In this case, it follows from Lemma 2.7 that

$$\max\{r_{1,i}, r_{1,j}\} \leq 2^6 \rho_3 |u_1 - u_2|.$$

Hence the inequalities in (3.1) and Lemma 2.7 show that

$$(3.6) \quad \begin{aligned} \ell(\gamma_{1,0}[u_1, u_2]) &\leq \rho_6 d_{D_1}(u_2) \\ &\leq (1 + 3\rho_2) \rho_6 r_{1,j} \\ &\leq 2^8 c \rho_2 \rho_3 |u_1 - u_2|. \end{aligned}$$

Finally, the proof of (3.2) follows from the inequalities (3.3), (3.4), (3.5) and (3.6) and the proof of the lemma is complete.  $\square$

For all  $x'_1$  and  $x'_2 \in D'_{1,0}$ , Lemma 2.7 implies that there exist  $s_1, s_2 \in \{1, \dots, k_1\}$  such that  $x_1 \in B_{1,s_1}$  and  $x_2 \in B_{1,s_2}$ . Let  $D_{1,1} = \cup_{i=s_1}^{s_2} B_{1,i}$  and let  $\alpha_0$  be a rectifiable curve joining  $x_1$  and  $x_2$  in  $D_{1,1}$ . Then it follows from the similar reasoning as in the proofs of Lemmas 2.7 and 2.9 that

**Lemma 3.5.** *If for all  $x' \in \alpha'_0$ ,  $\ell(\alpha'_0[x'_1, x']) \leq \mu_{10} d_{D'_{1,1}}(x')$  for some constant  $\mu_{10} \geq 1$ , then there exists a  $\rho_7$ -uniform domain*

$$D'_{2,1} = \bigcup_{i=1}^{k_3} B_{3,i} \subset D'_{1,1}$$

such that

- (1)  $x'_1, x'_2 \in \overline{D'_{2,1}}$ , the closure of  $D'_{2,1}$ ;
- (2) For each  $i \in \{1, \dots, k_3\}$ ,

$$\frac{1}{3 \cdot 2^{16} \mu_{10}^2} d_{D'_{1,1}}(x'_{3,i}) \leq r_{3,i} \leq \frac{1}{2^{16} \mu_{10}^2} d_{D'_{1,1}}(x'_{3,i});$$

- (3)  $|x'_{3,k_3} - x'_2| \leq \frac{1}{2^{16} \mu_{10}^2} d_{D'_{1,1}}(x'_{3,k_3})$ , where  $B_{3,i} = \mathbb{B}(x'_{3,i}, r_{3,i})$ ,  $x'_{3,i} \in \alpha'_0$ ,  $x'_{3,i} \notin B_{3,i-1}$  for all  $i \in \{2, \dots, k_3\}$  and  $\rho_7$  is a constant depending on  $\mu_{10}$ .

By Theorem A, Lemmas 2.9 and 3.5, we have

**Lemma 3.6.** *The domain  $D_{2,1} = f^{-1}(D'_{2,1})$  is a  $\rho_8$ -uniform, where  $\rho_8$  is a constant depending on  $\rho_5, \rho_7$  and  $K$ .*

**Lemma 3.7.** *For all  $u_1, u_2 \in \gamma_{1,0}$ ,  $\ell_{k_D}(\gamma_{1,0}[u_1, u_2]) \leq 2^{11}c^2\rho_2\rho_3k_D(u_1, u_2)$ .*

*Proof.* In view of Lemmas 3.3 and 3.4, we have

$$\begin{aligned} \ell_{k_D}(\gamma_{1,0}[u_1, u_2]) &= \int_{\gamma_{1,0}[u_1, u_2]} \frac{|dx|}{d_D(x)} \\ &\leq 2\rho_6 \log \left( 1 + \frac{2\ell(\gamma_{1,0}[u_1, u_2])}{d_D(u_1)} \right) \\ &\leq 2\rho_6 \log \left( 1 + 2^9 c\rho_2\rho_3 \frac{|u_1 - u_2|}{d_D(u_1)} \right) \\ &\leq 2^{11}c^2\rho_2\rho_3k_D(u_1, u_2) \end{aligned}$$

which concludes the proof of the lemma.  $\square$

For the convenience of the statement of our main lemma below, we list down the related constants that we use in the proof.

- (1)  $\rho_9 = 18\rho_2\rho_6\mu_9 \left( (2(1 + 3\mu_9)^{\mu_1(c)\mu_2^2} + 1) + 1 \right),$
- (2)  $\rho_{10} = (2^8 a^2 c^2 \rho_9^5 \mu_9)^n,$
- (3)  $\rho_{11} = \left( \frac{3\mu_2^2 K \mu_3(\rho_5, n) \rho_9^2 \mu_9}{\log(3/2)} \right)^{\mu_2^2},$
- (4)  $\rho_{12} = (\rho_{10}\rho_{11})^{14\mu_2},$
- (5)  $\rho_{13} = \rho_{10}\rho_{12}^2,$
- (6)  $\rho_{14} = \max \left\{ \rho_{13}^4, 4\rho_9 H(n, K, \mu_5(a), \varphi(a_1), \rho_{10}\rho_{11}, \rho_{13}^2) \right\},$
- (7)  $\rho_{15} = 2^{22} \cdot 3^{1+\mu_2} \rho_{14}^{30} (\rho_{11}\mu_2)^{\mu_2},$
- (8)  $\rho_{16} = \rho_7\rho_8 \max \left\{ \eta(\rho_{15}), (\rho_{10}\rho_{11})^{12} \right\},$
- (9)  $\rho_{17} = \psi_n^{-1} \left( \frac{\phi_n((\rho_2\rho_9\rho_{14}\rho_{16})^{72\mu_2^3})}{\mu_3(\rho_5, n)K} \right),$  and
- (10)  $\rho_{18} = (\rho_2\rho_{11})^{\rho_{17}}.$

**Lemma 3.8.** *For each  $z' \in \gamma'_{1,0}$ ,  $\text{diam}(\gamma'_{1,0}[z'_1, z']) < \rho_{18}d_{D'_1}(z')$ .*

*Proof.* Suppose on the contrary that there exists a point  $z' \in \gamma'_{1,0}$  such that

$$(3.7) \quad \text{diam}(\gamma'_{1,0}[z'_1, z']) \geq \rho_{18}d_{D'_1}(z').$$

Let  $z'_{1,0}$  be the first point in  $\gamma'_{1,0}$  along the direction from  $z'_1$  to  $z'_0$  satisfying

$$(3.8) \quad \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]) = \rho_{18}d_{D'_1}(z'_{1,0}),$$

and let  $z'_{0,0} \in \gamma'_{1,0}[z'_1, z'_{1,0}]$  such that

$$(3.9) \quad \text{diam}(\gamma'_{1,0}[z'_1, z'_{0,0}]) = \frac{1}{2} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

Since  $z_{1,0} \in \gamma_{1,0}$ , by Lemma 2.7, there exists an  $i_1 \in \{1, \dots, k_1\}$  such that  $z_{1,0} \in B_{1,i_1}$ .

**Claim 3.1.** *There exists  $v'_{1,0} \in \gamma'_{1,0}[z'_{0,0}, z'_{1,0}]$  such that*

$$|v'_{1,0} - x'_{1,i_1}| \geq \frac{1}{4} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]) \quad \text{and} \quad k_{D'_1}(x'_{1,i_1}, v'_{1,0}) \geq \log \frac{\rho_{18}}{4e^{\mu_2}}.$$

We first prove that there exists  $v'_{1,0} \in \gamma'_{1,0}[z'_{0,0}, z'_{1,0}]$  satisfying the first inequality. For each  $x' \in \gamma'_{1,0}[x'_{1,i_1}, z'_{1,0}]$ , we have

$$k_{D'_1}(x_{1,i_1}, x) \leq \int_{[x_{1,i_1}, x]} \frac{|dw|}{d_{D'_1}(w)} \leq \frac{1}{\rho_2 - 1}$$

which together with Theorem G show that

$$\max \left\{ \log \frac{d_{D'_1}(x'_{1,i_1})}{d_{D'_1}(x')}, \log \frac{d_{D'_1}(x')}{d_{D'_1}(x'_{1,i_1})}, \log \left( 1 + \frac{|x'_{1,i_1} - x'|}{d_{D'_1}(x'_{1,i_1})} \right) \right\} \leq k_{D'_1}(x'_{1,i_1}, x') < \mu_2,$$

whence

$$e^{-\mu_2} d_{D'_1}(x') \leq d_{D'_1}(x'_{1,i_1}) \leq e^{\mu_2} d_{D'_1}(x') \quad \text{and} \quad |x'_{1,i_1} - x'| \leq e^{\mu_2} d_{D'_1}(x'_{1,i_1}).$$

The last inequalities together with (3.8) imply that

$$(3.10) \quad d_{D'_1}(x'_{1,i_1}) \leq e^{\mu_2} d_{D'_1}(z'_{1,0}) = \frac{e^{\mu_2}}{\rho_{18}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}])$$

and so

$$|x'_{1,i_1} - x'| \leq \frac{e^{2\mu_2}}{\rho_{18}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

We conclude that

$$(3.11) \quad \text{diam}(\gamma'_{1,0}[x'_{1,i_1}, z'_{1,0}]) \leq \frac{2e^{2\mu_2}}{\rho_{18}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

Hence (3.9) implies

$$\text{diam}(\gamma'_{1,0}[z'_{0,0}, x'_{1,i_1}]) \geq \frac{2}{5} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

The existence of the needed point  $v'_{1,0} \in \gamma'_{1,0}[z'_{0,0}, x'_{1,i_1}]$  now follows.

The points  $v'_{1,0}$  determined as above must satisfy the second inequality in Claim 3.1. Indeed this is obvious from (3.10), the first inequality and the fact

$$k_{D'_1}(x'_{1,i_1}, v'_{1,0}) \geq \log \left( 1 + \frac{|v'_{1,0} - x'_{1,i_1}|}{d_{D'_1}(x'_{1,i_1})} \right).$$

**Claim 3.2.** *For all  $z \in \gamma_{1,0}[z_{0,0}, z_{1,0}]$ ,  $d_D(z) < \rho_{11} \ell(\gamma_{1,0}[z_1, z])$ .*

Suppose on the contrary that there exists some point  $z \in \gamma_{1,0}[z_{0,0}, z_{1,0}]$  such that

$$d_D(z) \geq \rho_{11} \ell(\gamma_{1,0}[z_1, z]).$$

Let  $x_{1,0}$  be the first point of  $\gamma_{1,0}[z_{0,0}, z_{1,0}]$  from  $z_{0,0}$  to  $z_{1,0}$  satisfying

$$d_D(x_{1,0}) \geq \rho_{11} \ell(\gamma_{1,0}[z_1, x_{1,0}]).$$

Then for every  $y \in \gamma_{1,0}[z_1, x_{1,0}]$ ,

$$k_D(y, x_{1,0}) \leq \int_{[y, x_{1,0}]} \frac{|dx|}{d_D(x)} \leq \frac{|y - x_{1,0}|}{(1 - \frac{1}{\rho_{11}})d_D(x_{1,0})} \leq \frac{1}{\rho_{11} - 1},$$

which together with Theorem G show that

$$\log \left( 1 + \frac{|y' - x'_{1,0}|}{d_{D'}(x'_{1,0})} \right) \leq k_{D'}(y', x'_{1,0}) \leq \mu_2(k_D(y, x_{1,0}))^{\frac{1}{\mu_2}} < \log \frac{6}{5}.$$

We conclude that

$$|y' - x'_{1,0}| < \frac{1}{5}d_{D'}(x'_{1,0})$$

which yields

$$\gamma'_{1,0}[z'_1, x'_{1,0}] \subset \mathbb{B}(x'_{1,0}, \frac{1}{5}d_{D'}(x'_{1,0})),$$

whence

$$\text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]) = 2\text{diam}(\gamma'_{1,0}[z'_1, z'_{0,0}]) < \frac{4}{5}d_{D'}(x'_{1,0}).$$

This implies  $z'_{1,0} \in \mathbb{B}(x'_{1,0}, \frac{4}{5}d_{D'}(x'_{1,0}))$ , and so

$$d_{D'}(z'_{1,0}) \geq d_{D'}(x'_{1,0}) - |z'_{1,0} - x'_{1,0}| \geq \frac{1}{5}d_{D'}(x'_{1,0})$$

showing that

$$\text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]) \leq 4d_{D'}(z'_{1,0}).$$

Therefore we infer from (3.8) that

$$(3.12) \quad d_{D'_1}(z'_{1,0}) \leq \frac{4}{\rho_{18}}d_{D'}(z'_{1,0}).$$

Concerning the relation of the curve  $\gamma_{1,0}[z_1, z_{1,0}]$  and the sphere  $\mathbb{S}(z_{1,0}, \frac{1}{\rho_{11}}d_{D_1}(z_{1,0}))$ , we have

**Proposition 3.1.**  $\gamma_{1,0}[z_1, z_{1,0}] \cap \mathbb{S}(z_{1,0}, \frac{1}{\rho_{11}}d_{D_1}(z_{1,0})) \neq \emptyset$ .

Suppose on the contrary that  $\gamma_{1,0}[z_1, z_{1,0}] \cap \mathbb{S}(z_{1,0}, \frac{1}{\rho_{11}}d_{D_1}(z_{1,0})) = \emptyset$ . Then for each  $y \in \gamma_{1,0}[z_1, z_{1,0}]$ ,

$$(3.13) \quad k_{D_1}(y, z_{1,0}) \leq \int_{[y, z_{1,0}]} \frac{|dx|}{d_{D_1}(x)} \leq \frac{|y - z_{1,0}|}{(1 - \frac{1}{\rho_{11}})d_{D_1}(z_{1,0})} \leq \frac{1}{\rho_{11} - 1}.$$

Let  $y'_1 \in \gamma'_{1,0}[z'_1, z'_{1,0}]$  such that  $|y'_1 - z'_{1,0}| \geq \frac{1}{2}\text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}])$ . Then (3.8) implies that

$$k_{D'_1}(y'_1, z'_{1,0}) \geq \log \left( 1 + \frac{|y'_1 - z'_{1,0}|}{d_{D'_1}(z'_{1,0})} \right) \geq \log \left( 1 + \frac{\rho_{18}}{2} \right).$$

By Theorem G and (3.13)

$$\log \left( 1 + \frac{\rho_{18}}{2} \right) \leq k_{D'_1}(y'_1, z'_{1,0}) \leq \mu_2(k_{D_1}(y_1, z_{1,0}))^{\frac{1}{\mu_2}} \leq \mu_2 \left( \frac{1}{\rho_{11} - 1} \right)^{\frac{1}{\mu_2}}$$

which is a contradiction, and therefore, the proof of the proposition is complete.

Now, we continue the proof of Claim 3.2. The rest of the discussion is divided into two cases.

**Case 3.1.**  $d_D(z_{1,0}) < 2cd_{D_1}(z_{1,0})$ .

By Proposition 3.1, we take  $z_{1,1} \in \gamma_{1,0}[z_1, z_{1,0}] \cap \mathbb{S}(z_{1,0}, \frac{1}{\rho_{11}}d_{D_1}(z_{1,0}))$ . Then

$$(3.14) \quad k_D(z_{1,1}, z_{1,0}) \geq \log \left( 1 + \frac{|z_{1,1} - z_{1,0}|}{d_D(z_{1,0})} \right) \geq \log \left( 1 + \frac{1}{2c\rho_{11}} \right).$$

Since

$$k_{D_1}(z_{1,1}, z_{1,0}) \leq \int_{[z_{1,1}, z_{1,0}]} \frac{|dx|}{d_{D_1}(x)} \leq \frac{1}{\rho_{11} - 1},$$

Theorem G gives

$$\log \left( 1 + \frac{|z'_{1,1} - z'_{1,0}|}{d_{D'_1}(z'_{1,0})} \right) \leq k_{D'_1}(z'_{1,1}, z'_{1,0}) \leq \mu_2(k_{D_1}(z_{1,1}, z_{1,0}))^{\frac{1}{\mu_2}} < \log \frac{6}{5}$$

which shows that

$$|z'_{1,1} - z'_{1,0}| < \frac{1}{5}d_{D'_1}(z'_{1,0}).$$

Hence the inequalities (3.12), (3.14), and Theorem G yield

$$\left( \frac{1}{\mu_2} \log \left( 1 + \frac{1}{2c\rho_{11}} \right) \right)^{\mu_2} \leq k_{D'}(z'_{1,1}, z'_{1,0}) \leq \int_{[z'_{1,1}, z'_{1,0}]} \frac{|dw'|}{d_{D'}(w')} \leq \frac{1}{\rho_{18} - 1}.$$

This is a contradiction.

**Case 3.2.**  $d_D(z_{1,0}) \geq 2cd_{D_1}(z_{1,0})$ .

We let  $z'_{1,2} \in \gamma'_{1,0}[z'_1, z'_{1,0}]$  such that

$$(3.15) \quad |z'_{1,2} - z'_{1,0}| \geq \frac{1}{2} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

Let  $B_1 = \mathbb{B}(z_{1,2}, \frac{1}{4}d_{D_1}(z_{1,0}))$  and  $B_2 = \mathbb{B}(z_{1,0}, \frac{1}{\rho_{11}}d_{D_1}(z_{1,0}))$ . Then, since (3.1) yields

$$(3.16) \quad |z_{1,2} - z_{1,0}| \leq \ell(\gamma_{1,0}[z_{1,2}, z_{1,0}]) \leq \rho_6 d_{D_1}(z_{1,0}),$$

we obtain from the assumption “ $d_D(z_{1,0}) \geq 2cd_{D_1}(z_{1,0})$ ” that

$$(3.17) \quad B_1 \cup B_2 \subset \mathbb{B}(z_{1,0}, d_D(z_{1,0})).$$

We infer from (3.8) and (3.15) that

$$k_{D'_1}(z'_{1,2}, z'_{1,0}) \geq \log \left( 1 + \frac{|z'_{1,2} - z'_{1,0}|}{d_{D'_1}(z'_{1,0})} \right) \geq \log \frac{\rho_{18}}{2},$$

whence Theorem G shows that

$$k_{D_1}(z_{1,2}, z_{1,0}) > 1.$$

Then  $B_1 \cap B_2 = \emptyset$ , because otherwise  $z_{1,2} \in \mathbb{B}(z_{1,0}, \frac{1}{2}d_{D_1}(z_{1,0}))$  implies that  $k_{D_1}(z_{1,2}, z_{1,0}) \leq 1$ .



Since for every  $y \in \mathbb{S}(z_{1,0}, \frac{1}{\rho_{11}}d_{D_1}(z_{1,0}))$ ,

$$k_{D_1}(y, z_{1,0}) \leq \int_{[y, z_{1,0}]} \frac{|dw|}{d_{D_1}(w)} \leq \frac{1}{\rho_{11} - 1},$$

we see from Theorem G again that

$$(3.18) \quad k_{D'_1}(y', z'_{1,0}) \leq \mu_2(k_{D_1}(y, z_{1,0}))^{\frac{1}{\mu_2}} < \frac{4}{9} \log \frac{3}{2}.$$

Also, we see that for all  $z' \in \mathbb{B}(z'_{1,0}, d_{D'_1}(z'_{1,0})) \setminus \mathbb{B}(z'_{1,0}, \frac{1}{2}d_{D'_1}(z'_{1,0}))$ ,

$$k_{D'_1}(z', z'_{1,0}) \geq \log \left( 1 + \frac{|z'_{1,0} - z'|}{d_{D'_1}(z'_{1,0})} \right) \geq \log \frac{3}{2}.$$

Hence (3.18) implies

$$(3.19) \quad f(B_2) \subset \mathbb{B}(z'_{1,0}, \frac{1}{2}d_{D'_1}(z'_{1,0}))$$

whence (3.8) and (3.15) show that

$$(3.20) \quad \begin{aligned} \text{dist}(z'_{1,2}, f(B_2)) &\geq |z'_{1,2} - z'_{1,0}| - \text{diam}(f(B_2)) \\ &\geq \frac{\rho_{18} - 2}{2\rho_{18}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]). \end{aligned}$$

We get from (3.17) and (3.20) that there exists a simply connected domain  $G_1 \subset B_1$  such that

- (1)  $z_{1,2} \in G_1$ ;
- (2)  $\text{diam}(G_1) \geq \frac{1}{4}d_{D_1}(z_{1,0})$ ; and
- (3)  $\text{dist}(f(G_1), f(B_2)) \geq \frac{1}{4}\text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}])$ .

We deduce from (3.16), (3.17), Theorems H and I that

$$\text{Mod}(G_1, B_2; \mathbb{B}(z_{1,0}, d_D(z_{1,0}))) \geq \frac{1}{\mu_3(n, \rho)} \text{Mod}(G_1, B_2; \mathbb{R}^n) \geq \frac{1}{\mu_3(n, \rho)} \phi_n\left(\frac{4}{7}c\rho_{11}\right),$$

where  $\rho = \frac{\pi}{2}$ . But we get from (3.8), (3.19) and Theorem H that

$$\text{Mod}\left(f(G_1), f(B_2); f(\mathbb{B}(z_{1,0}, d_D(z_{1,0})))\right) \leq \text{Mod}(f(G_1), f(B_2); \mathbb{R}^n) \leq \psi_n\left(\frac{1}{4}\rho_{18}\right),$$

which implies

$$\phi_n\left(\frac{4}{7}c\rho_{11}\right) \leq \mu_3(n, \rho)K\psi_n\left(\frac{1}{4}\rho_{18}\right) < \mu_3(n, \rho)K\psi_n(\rho_{17}) < \phi_n((\rho_2\rho_9\rho_{14}\rho_{16})^{72\mu_2^3}).$$

This is clearly a contradiction. We complete the proof of Claim 3.2.

The following result easily follows from (3.1) and Claim 3.2.

**Corollary 3.1.** *For each  $z \in \gamma_{1,0}[z_{0,0}, z_{1,0}]$ ,  $d_D(z) < \rho_6\rho_{11}d_{D_1}(z)$ .*

**Claim 3.3.** *For all  $x_1, x_2 \in \gamma_{1,0}[z_{0,0}, z_{1,0}]$ , we have*

$$k_{D_1}(x_1, x_2) \leq 2^9 c^2 \rho_2 \rho_{11} k_D(x_1, x_2).$$

If  $|x_2 - x_1| \leq \frac{1}{2}d_{D_1}(x_1)$ , then Corollary 3.1 implies that

$$\begin{aligned}
k_{D_1}(x_1, x_2) &\leq \int_{[x_1, x_2]} \frac{|dx|}{d_{D_1}(x)} \\
&\leq \frac{2|x_2 - x_1|}{d_{D_1}(x_1)} \\
&\leq 3 \log \left( 1 + \frac{|x_2 - x_1|}{d_{D_1}(x_1)} \right) \\
&\leq 3\rho_6\rho_{11} \log \left( 1 + \frac{|x_2 - x_1|}{d_D(x_1)} \right) \\
&\leq 3\rho_6\rho_{11} k_D(x_1, x_2).
\end{aligned}$$

If  $|x_2 - x_1| > \frac{1}{2}d_{D_1}(x_1)$ , then it follows from Corollary 3.1, Lemmas 3.3 and 3.4 that

$$\begin{aligned}
k_{D_1}(x_1, x_2) &\leq \int_{\gamma_{1,0}[x_1, x_2]} \frac{|dx|}{d_{D_1}(x)} \\
&\leq 2\rho_6 \log \left( 1 + \frac{2\ell(\gamma_{1,0}[x_1, x_2])}{d_{D_1}(x_1)} \right) \\
&\leq 2\rho_6 \log \left( 1 + 2^9 c\rho_2\rho_3 \frac{|x_2 - x_1|}{d_{D_1}(x_1)} \right) \\
&\leq 2\rho_6 \log \left( 1 + \frac{2^{12}}{7} c^2 \rho_2\rho_3\rho_{11} \frac{|x_2 - x_1|}{d_D(x_1)} \right) \\
&\leq 2^9 c^2 \rho_2\rho_{11} k_D(x_1, x_2).
\end{aligned}$$

The proof of Claim 3.3 is complete.

Let  $\alpha'_{1,0}$  be an arc in  $D'_{1,0}$  with endpoints  $w'_1, w'_2 \in D'_{1,0}$ . We introduce the following concept.

***QH-Condition.*** We say that  $\alpha'_{1,0}$  satisfies *QH-Condition* if the following hold.

- (1)  $k_{D'}(w'_1, w'_2) \geq (\rho_{10}\rho_{11})^7$ ;
- (2) There exist successive points  $\eta'_0 (= w'_1), \eta'_1, \dots, \eta'_{\rho_{10}} (= w'_2)$  in  $\alpha'_{1,0}$  and a constant  $\mu_{11} \in [1, (\rho_{10}\rho_{11})^3]$  such that for all  $i, j \in \{0, \dots, \rho_{10} - 1\}$ ,

$$\frac{1}{\mu_{11}} k_{D'}(\eta'_j, \eta'_{j+1}) \leq k_{D'}(\eta'_i, \eta'_{i+1}) \leq \mu_{11} k_{D'}(\eta'_j, \eta'_{j+1}).$$

Let  $w'_0 \in \alpha'_{1,0}$  such that

$$(3.21) \quad d_{D'_{1,0}}(w'_0) = \inf_{z' \in \alpha'_{1,0}} d_{D'_{1,0}}(z'),$$

and for each  $i \in \{0, \dots, \rho_{10} - 1\}$ , we let  $u'_i \in \alpha'_{1,0}[\eta'_i, \eta'_{i+1}]$  such that

$$k_{D'}(u'_i, \eta'_{i+1}) = \frac{1}{4\mu_2^2\mu_{11}} k_{D'}(\eta'_i, \eta'_{i+1}).$$

**Proposition 3.2.** *Suppose  $\alpha'_{1,0}$  satisfies QH-Condition. Then for every  $i \in \{0, 1, \dots, \rho_{10} - 1\}$ ,*

$$\min\{d_{D'}(u'_i), d_{D'}(\eta'_{i+1})\} < \text{diam}(\alpha'_{1,0}).$$

We prove the proposition by contradiction. Suppose there exists a  $t \in \{0, 1, \dots, \rho_{10} - 1\}$  such that

$$\min\{d_{D'}(u'_t), d_{D'}(\eta'_{t+1})\} \geq \text{diam}(\alpha'_{1,0}).$$

Then

$$[u'_t, \eta'_{t+1}] \subset \overline{\mathbb{B}}(u'_t, \frac{1}{2}d_{D'}(u'_t)) \cup \overline{\mathbb{B}}(\eta'_{t+1}, \frac{1}{2}d_{D'}(\eta'_{t+1})),$$

which implies

$$k_{D'}(u'_t, \eta'_{t+1}) \leq 2$$

so that

$$\begin{aligned} k_{D'}(w'_1, w'_2) &\leq \sum_{i=0}^{\rho_{10}-1} k_{D'}(\eta'_i, \eta'_{i+1}) \\ &\leq 4\mu_2^2 \mu_{11}^2 \sum_{i=0}^{\rho_{10}-1} k_{D'}(u'_t, \eta'_{t+1}) \\ &\leq 8\rho_{10}\mu_2^2 \mu_{11}^2, \end{aligned}$$

which contradicts the assumption that “ $k_{D'}(w'_1, w'_2) \geq (\rho_{10}\rho_{11})^7$ ”, and thus the proof of Proposition 3.2 is complete.

**Proposition 3.3.** *Suppose  $\alpha'_{1,0}$  satisfies QH-Condition. Then*

$$k_{D'}(w'_1, w'_2) \leq 48a^2 \rho_{10} \mu_2^2 \mu_{11}^2 \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right).$$

To prove this proposition, we let  $\beta'_i$  be an  $a$ -cone curve joining  $u'_i$  and  $\eta'_{i+1}$  in  $D'$  for each  $i \in \{0, \dots, \rho_{10} - 1\}$ . This can be done because  $D'$  is an  $a$ -John domain.

In the case where there exists some  $s \in \{0, \dots, \rho_{10} - 1\}$  such that

$$\ell(\beta'_s) \leq \text{diam}(\alpha'_{1,0}),$$

we deduce from Lemma 2.6 that

$$\begin{aligned} k_{D'}(u'_s, \eta'_{s+1}) &\leq \int_{\beta'_s[u'_s, \eta'_{s+1}]} \frac{|dx|}{d_{D'}(x)} \\ &\leq 8a \log \left( 1 + \frac{\ell(\beta'_s)}{d_{D'}(w'_0)} \right) \\ &\leq 8a \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right), \end{aligned}$$

whence

$$\begin{aligned}
 (3.22) \quad k_{D'}(w'_1, w'_2) &\leq \sum_{i=0}^{\rho_{10}-1} k_{D'}(\eta'_i, \eta'_{i+1}) \\
 &\leq 4\mu_2^2 \mu_{11}^2 \sum_{i=0}^{\rho_{10}-1} k_{D'}(u'_s, \eta'_{s+1}) \\
 &\leq 32a\rho_{10}\mu_2^2 \mu_{11}^2 \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right).
 \end{aligned}$$

In the remaining case, if for each  $i \in \{0, \dots, \rho_{10} - 1\}$

$$\ell(\beta'_i) > \text{diam}(\alpha'_{1,0})$$

holds, then it follows from Proposition 3.2 that for each  $i \in \{0, 1, \dots, \rho_{10} - 1\}$ , there exists  $v'_i \in \beta'_i$  such that

$$\frac{1}{2a} \text{diam}(\alpha'_{1,0}) \leq d_{D'}(v'_i) < \text{diam}(\alpha'_{1,0}).$$

We easily obtain that for all  $i \in \{0, \dots, \rho_{10} - 1\}$ ,

$$\mathbb{B}(v'_i, \frac{1}{2}d_{D'}(v'_i)) \subset \mathbb{B}(\eta'_0, (a + \frac{3}{2})\text{diam}(\alpha'_{1,0})).$$

Then there exist  $p \neq q \in \{0, \dots, \rho_{10} - 1\}$  such that

$$(3.23) \quad \mathbb{B}(v'_p, \frac{1}{2}d_{D'}(v'_p)) \cap \mathbb{B}(v'_q, \frac{1}{2}d_{D'}(v'_q)) \neq \emptyset,$$

because otherwise,

$$\begin{aligned}
 \left(a + \frac{3}{2}\right)^n \text{Vol}(\mathbb{B}(\eta'_0, \text{diam}(\alpha'_{1,0}))) &= \text{Vol}(\mathbb{B}(\eta'_0, (a + \frac{3}{2})\text{diam}(\alpha'_{1,0}))) \\
 &> \sum_{i=0}^{\rho_{10}-1} \text{Vol}(\mathbb{B}^n(v'_i, \frac{1}{2}d_{D'}(v'_i))) \\
 &\geq \left(\frac{1}{4a}\right)^n \rho_{10} \text{Vol}(\mathbb{B}^n(\eta'_0, \text{diam}(\alpha'_{1,0}))),
 \end{aligned}$$

where “Vol” denotes the volume. This is clearly a contradiction.

We divide the rest of the arguments into four cases:

- (1)  $\ell(\beta'_p[v'_p, \eta'_{p+1}]) \leq \frac{1}{2}\ell(\beta'_p)$  and  $\ell(\beta'_q[v'_q, u'_q]) \leq \frac{1}{2}\ell(\beta'_q)$ ;
- (2)  $\ell(\beta'_p[v'_p, u'_p]) \leq \frac{1}{2}\ell(\beta'_p)$  and  $\ell(\beta'_q[v'_q, u'_q]) \leq \frac{1}{2}\ell(\beta'_q)$ ;
- (3)  $\ell(\beta'_p[v'_p, \eta'_{p+1}]) \leq \frac{1}{2}\ell(\beta'_p)$  and  $\ell(\beta'_q[v'_q, \eta'_{q+1}]) \leq \frac{1}{2}\ell(\beta'_q)$ ;
- (4)  $\ell(\beta'_p[v'_p, u'_p]) \leq \frac{1}{2}\ell(\beta'_p)$  and  $\ell(\beta'_q[v'_q, \eta'_{q+1}]) \leq \frac{1}{2}\ell(\beta'_q)$ .

It suffices to discuss the first case since the discussions for the remaining three cases are similar.

By Lemma 2.6, we see that

$$\begin{aligned}
k_{D'}(v'_p, \eta'_{p+1}) &\leq \ell_{k_{D'}}(\beta'_p(v'_p, \eta'_{p+1})) \\
&\leq 4a \log \left( 1 + \frac{\ell(\beta'_p(v'_p, \eta'_{p+1}))}{d_{D'}(\eta'_{p+1})} \right) \\
&\leq 4a^2 \log \left( 1 + \frac{d_{D'}(v'_p)}{d_{D'}(w'_0)} \right) \\
&\leq 4a^2 \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right), \\
k_{D'}(u'_q, v'_q) &\leq 4a^2 \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right)
\end{aligned}$$

and by (3.23),

$$k_{D'}(v'_p, v'_q) \leq 2.$$

Hence we conclude from Proposition 3.2 and (3.21) that

$$\begin{aligned}
k_{D'}(\eta'_{p+1}, u'_q) &\leq k_{D'}(\eta'_{p+1}, v'_p) + k_{D'}(v'_p, v'_q) + k_{D'}(v'_q, u'_q) \\
&\leq 8a^2 \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right) + 2 \\
&\leq 12a^2 \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3.24) \quad k_{D'}(w'_1, w'_2) &\leq \sum_{i=0}^{\rho_{10}-1} k_{D'}(\eta'_i, \eta'_{i+1}) \\
&\leq 4\mu_2^2 \mu_{11}^2 \sum_{i=0}^{\rho_{10}-1} k_{D'}(\eta'_{p+1}, u'_p) \\
&\leq 48a^2 \rho_{10} \mu_2^2 \mu_{11}^2 \log \left( 1 + \frac{\text{diam}(\alpha'_{1,0})}{d_{D'}(w'_0)} \right).
\end{aligned}$$

The combination of (3.22) and (3.24) completes the proof of Proposition 3.3.

Suppose that  $v'_{1,0} \in \gamma'_{1,0}$  satisfies Claim 3.1 and  $\gamma'_{2,0}$  is a quasihyperbolic geodesic joining  $x'_{1,i_1}$  and  $v'_{1,0}$  in  $D'_{1,0}$ . We recall that  $D_{1,0}$  is a  $\rho_5$ -uniform domain (see Lemma 2.9). Then

**Claim 3.4.** *For  $z \in \gamma_{2,0}$ , we have  $|w - z| < \rho_9 d_{D_{1,0}}(z)$  for every  $w \in \gamma_{2,0}[v_{1,0}, z]$ .*

To prove this claim, we let  $w_{0,0} \in \gamma_{2,0}$  such that

$$d_{D_{1,0}}(w_{0,0}) = \sup_{p \in \gamma_{2,0}} d_{D_{1,0}}(p).$$

If  $w_{0,0} \in \gamma_{2,0}[z, x_{1,i_1}]$ , then Corollary 2.3 shows

$$|w - z| \leq \mu_9 d_{D_{1,0}}(z).$$

If  $w_{0,0} \in \gamma_{2,0}[v_{1,0}, z]$ , then it follows from Corollary 2.3 that

$$(3.25) \quad |x_{1,i_1} - z| \leq \mu_9 d_{D_{1,0}}(z).$$

If  $|x_{1,i_1} - z| \leq \frac{1}{2}d_{D_{1,0}}(x_{1,i_1})$ , then it is obvious that

$$d_{D_{1,0}}(z) \geq \frac{1}{2}d_{D_{1,0}}(x_{1,i_1}).$$

If  $|x_{1,i_1} - z| > \frac{1}{2}d_{D_{1,0}}(x_{1,i_1})$ , it follows from (3.25) that

$$d_{D_{1,0}}(z) \geq \frac{1}{2\mu_9}d_{D_{1,0}}(x_{1,i_1}).$$

Thus, we have

$$(3.26) \quad d_{D_{1,0}}(z) \geq \frac{1}{2\mu_9}d_{D_{1,0}}(x_{1,i_1}).$$

It follows from Claim 3.1 and Theorem G that

$$k_{D_{1,0}}(x_{1,i_1}, v_{1,0}) \geq 1$$

whence

$$\max\{d_{D_{1,0}}(v_{1,0}), d_{D_{1,0}}(x_{1,i_1})\} \leq 2|x_{1,i_1} - v_{1,0}|.$$

We get from (3.25), (3.26), Lemmas 2.3, 2.7 and 3.2 that

$$\begin{aligned} |w - z| &\leq |w - v_{1,0}| + |v_{1,0} - x_{1,i_1}| + |x_{1,i_1} - z| \\ &\leq 3(2(1 + 3\mu_9)^{\mu_1(c)\mu_2^2} + 1)|v_{1,0} - x_{1,i_1}| + |x_{1,i_1} - z| \\ &\leq 3\rho_6(2(1 + 3\mu_9)^{\mu_1(c)\mu_2^2} + 1)d_{D_1}(x_{1,i_1}) + \mu_9 d_{D_{1,0}}(z) \\ &\leq 9\rho_2\rho_6(2(1 + 3\mu_9)^{\mu_1(c)\mu_2^2} + 1)d_{D_{1,0}}(x_{1,i_1}) + \mu_9 d_{D_{1,0}}(z) \\ &\leq 18\rho_2\rho_6\mu_9((2(1 + 3\mu_9)^{\mu_1(c)\mu_2^2} + 1) + 1)d_{D_{1,0}}(z), \end{aligned}$$

from which the proof of Claim 3.4 is complete.

**Claim 3.5.** *For each  $z \in \gamma_{2,0}$ , we have*

- (1)  $(\rho_2 - 1)d_{D_{1,0}}(z) \leq d_{D_1}(z) \leq d_D(z) \leq 2^9(3\rho_2 + 1)\rho_3\rho_6\rho_9\rho_{11}d_{D_{1,0}}(z);$
- (2)  $d_{D'_1}(z') \leq (2^{13}c\rho_2\rho_3\rho_9\rho_{11}\mu_2)^{\mu_2}d_{D'_{1,0}}(z').$

We first prove Claim 3.5(1). Since  $\gamma_{2,0} \subset D_{1,0}$ , it follows from Lemma 3.1 that for each  $z \in \gamma_{2,0}$ ,  $d_{D_1}(z) \geq (\rho_2 - 1)d_{D_{1,0}}(z)$ .

It follows from Claim 3.4 and the similar reasoning as in the proof of (3.26) that

$$d_{D_{1,0}}(z) \geq \frac{1}{\rho_9}|v_{1,0} - z|$$

and

$$d_{D_{1,0}}(z) \geq \frac{1}{2\rho_9}d_{D_{1,0}}(v_{1,0}).$$

Then we deduce from Lemma 3.1 and Corollary 3.1 that

$$d_{D_{1,0}}(z) \geq \frac{1}{2^8(3\rho_2 + 1)\rho_3\rho_9}d_{D_1}(v_{1,0}) \geq \frac{1}{2^8c(3\rho_2 + 1)\rho_3\rho_6\rho_9\rho_{11}}d_D(v_{1,0}).$$

If  $|v_{1,0} - z| > \frac{1}{2}d_D(z)$ , then

$$d_{D_{1,0}}(z) \geq \frac{1}{\rho_9}|v_{1,0} - z| \geq \frac{1}{2\rho_9}d_D(z).$$

If  $|v_{1,0} - z| \leq \frac{1}{2}d_D(z)$ , then

$$\frac{1}{2}d_D(z) \leq d_D(v_{1,0}).$$

Thus,

$$d_D(z) \leq 2^9(3\rho_2 + 1)\rho_3\rho_6\rho_9\rho_{11}d_{D_{1,0}}(z).$$

We finish the proof of Claim 3.5(1).

The proof of Claim 3.5(2) easily follows from a similar argument as in that of Lemma 3.1(2). Hence Claim 3.5 holds.

**Claim 3.6.** *If  $k_{D'_{1,0}}(w'_1, w'_2) \geq \rho_{10}\rho_{11}$  for  $w'_1, w'_2 \in \gamma'_{2,0}$ , then*

$$k_{D'}(w'_1, w'_2) \geq \frac{1}{2^{12}\rho_9\rho_{11}\mu_2^2\mu_9}k_{D'_{1,0}}(w'_1, w'_2).$$

Let's now prove the claim. Since by Theorem G

$$(3.27) \quad k_{D_{1,0}}(w_1, w_2) \geq \frac{1}{\mu_2}k_{D'_{1,0}}(w'_1, w'_2),$$

we obtain from Corollary 2.3 and Claim 3.5 that

$$\begin{aligned} k_{D_{1,0}}(w_1, w_2) &\leq \mu_9 \log \left( 1 + \frac{|w_1 - w_2|}{\min\{d_{D_{1,0}}(w_1), d_{D_{1,0}}(w_2)\}} \right) \\ &\leq \mu_9 \log \left( 1 + 2^{12}c\rho_2\rho_3\rho_9\rho_{11} \frac{|w_1 - w_2|}{\min\{d_D(w_1), d_D(w_2)\}} \right) \end{aligned}$$

which together with (3.27) imply

$$\frac{|w_1 - w_2|}{\min\{d_D(w_1), d_D(w_2)\}} \geq 1.$$

Hence

$$\begin{aligned} (3.28) \quad k_{D_{1,0}}(w_1, w_2) &\leq 2^{12}\rho_9\rho_{11}\mu_9 \log \left( 1 + \frac{|w_1 - w_2|}{\min\{d_D(w_1), d_D(w_2)\}} \right) \\ &\leq 2^{12}\rho_9\rho_{11}\mu_9 k_D(w_1, w_2), \end{aligned}$$

whence  $k_D(w_1, w_2) > \mu_2$ . Therefore, (3.27), (3.28) and Theorem G yield

$$k_{D'}(w'_1, w'_2) \geq \frac{1}{\mu_2}k_D(w_1, w_2) \geq \frac{1}{2^{12}\rho_9\rho_{11}\mu_2^2\mu_9}k_{D'_{1,0}}(w'_1, w'_2),$$

from which Claim 3.6 follows.

As the first application of Claim 3.6, we have

**Claim 3.7.** *If  $k_{D'_{1,0}}(v'_1, v'_2) \geq \rho_{12}$  for  $v'_1, v'_2 \in \gamma'_{2,0}$ , then  $\gamma'_{2,0}[v'_1, v'_2]$  satisfies the QH-Condition with the constant  $\mu_{11} = 2^{12}\rho_9\rho_{11}\mu_2^2\mu_9$ .*

In order to prove the claim, we let  $\zeta'_1 = v'_1, \dots, \zeta'_{\rho_{10}+1} = v'_2$  in  $\gamma'_{2,0}[v'_1, v'_2]$  such that for all  $i \in \{1, \dots, \rho_{10}\}$ ,

$$k_{D'_{1,0}}(\zeta'_i, \zeta'_{i+1}) = \frac{1}{\rho_{10}} k_{D'_{1,0}}(v'_1, v'_2).$$

By Claim 3.6, we have

$$k_{D'}(v'_1, v'_2) \geq \frac{1}{2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9} k_{D'_{1,0}}(v'_1, v'_2) > (\rho_{10} \rho_{11})^{10}.$$

Since  $k_{D'_{1,0}}(\zeta'_i, \zeta'_{i+1}) \geq \frac{\rho_{12}}{\rho_{10}}$  for each  $i \in \{1, \dots, \rho_{10}\}$ , we see that for all  $i, j \in \{1, \dots, \rho_{10}\}$ , Claim 3.6 shows again that

$$\begin{aligned} k_{D'}(\zeta'_i, \zeta'_{i+1}) &\geq \frac{1}{2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9} k_{D'_{1,0}}(\zeta'_i, \zeta'_{i+1}) \\ &\geq \frac{1}{2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9} k_{D'}(\zeta'_j, \zeta'_{j+1}) \\ &\geq \frac{1}{(\rho_{10} \rho_{11})^3} k_{D'}(\zeta'_j, \zeta'_{j+1}). \end{aligned}$$

This completes the proof of Claim 3.7.

**Claim 3.8.**  $\gamma'_{1,0}[v'_{1,0}, x'_{1,i_1}]$  satisfies the QH-Condition with the constant

$$\mu_{11} = 2^{11} c^2 \rho_2 \rho_3 \mu_2^2.$$

For the proof of this claim, we let  $\theta_1 = v_{1,0}, \dots, \theta_{\rho_{10}+1} = x_{1,i_1}$  in  $\gamma_{1,0}[v_{1,0}, x_{1,i_1}]$  such that for each  $i \in \{1, \dots, \rho_{10}\}$ ,

$$\ell_{k_D}(\gamma_{1,0}[\theta_i, \theta_{i+1}]) = \frac{1}{\rho_{10}} \ell_{k_D}(\gamma_{1,0}[v_{1,0}, x_{1,i_1}]).$$

It follows from Claims 3.1 and (3.6) that

$$(3.29) \quad k_{D'}(v'_{1,0}, x'_{1,i_1}) \geq \frac{1}{2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9} k_{D'_{1,0}}(v'_{1,0}, x'_{1,i_1}) \geq \frac{1}{2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9} \log \frac{\rho_{18}}{4e^{\mu_2}},$$

whence Theorem G yields

$$k_D(v_{1,0}, x_{1,i_1}) \geq \frac{1}{\mu_2} k_{D'}(v'_{1,0}, x'_{1,i_1}) \geq \frac{1}{2^{12} \rho_9 \rho_{11} \mu_2^3 \mu_9} \log \frac{\rho_{18}}{4e^{\mu_2}}.$$

By Lemma 3.7, we know that for each  $i \in \{1, \dots, \rho_{10}\}$ ,

$$k_D(\theta_i, \theta_{i+1}) \geq \frac{1}{2^{11} c^2 \rho_2 \rho_3} \ell_{k_D}(\gamma_{1,0}[\theta_i, \theta_{i+1}]) \geq \frac{1}{2^{11} c^2 \rho_2 \rho_3 \rho_{10}} k_D(v_{1,0}, x_{1,i_1}) > \mu_2.$$

Hence for  $i, j \in \{1, \dots, \rho_{10}\}$ , Theorem G and Lemma 3.7 give

$$\begin{aligned} k_{D'}(\theta'_i, \theta'_{i+1}) &\leq \mu_2 k_D(\theta_i, \theta_{i+1}) \\ &\leq \mu_2 \ell_{k_D}(\gamma_{1,0}[\theta_j, \theta_{j+1}]) \\ &\leq 2^{11} c^2 \rho_2 \rho_3 \mu_2 k_D(\theta_j, \theta_{j+1}) \\ &\leq 2^{11} c^2 \rho_2 \rho_3 \mu_2^2 k_{D'}(\theta'_j, \theta'_{j+1}) \\ &\leq (\rho_{10} \rho_{11})^3 k_{D'}(\theta'_j, \theta'_{j+1}) \end{aligned}$$



which together with (3.29) complete the proof of Claim 3.8.

**Claim 3.9.**  $k_{D'_{1,0}}(x'_{1,i_1}, v'_{1,0}) \leq \rho_{10}^2 \rho_{11}^2 \rho_{17}$ .

To prove this claim, we obtain from Claim 3.1 and the inequalities (3.10) and (3.11) that

$$k_{D'_{1,0}}(x'_{1,i_1}, v'_{1,0}) \geq \log \left( 1 + \frac{|x'_{1,i_1} - v'_{1,0}|}{d_{D'_{1,0}}(x'_{1,i_1})} \right) > \log \left( 1 + \frac{\rho_{18}}{4e^{\mu_2}} \right) > (\rho_{10}\rho_{11})^8$$

and

$$\text{diam}(\gamma'_{1,0}[x'_{1,i_1}, v'_{1,0}]) \leq \left( 1 + \frac{2e^{2\mu_2}}{\rho_{18}} \right) \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

It follows from Claims 3.6, 3.8 and Proposition 3.3 that

$$\begin{aligned} (3.30) \quad k_{D'_{1,0}}(x'_{1,i_1}, v'_{1,0}) &\leq 2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9 k_{D'}(x'_{1,i_1}, v'_{1,0}) \\ &\leq 2^{18} a^2 \rho_9 \rho_{10} \rho_{11} \mu_2^4 \mu_9 \mu_{11}^2 \log \left( 1 + \frac{\text{diam}(\gamma'_{1,0}[x'_{1,i_1}, v'_{1,0}])}{d_{D'}(w'_0)} \right) \\ &\leq \rho_{10}^2 \rho_{11} \log \left( 1 + \left( 1 + \frac{2e^{2\mu_2}}{\rho_{18}} \right) \frac{\text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}])}{d_{D'}(w'_0)} \right), \end{aligned}$$

where  $w'_0 \in \gamma'_{1,0}[x'_{1,i_1}, v'_{1,0}]$  satisfies

$$d_{D'_{1,0}}(w'_0) = \inf \{ d_{D'_{1,0}}(z') : z' \in \gamma'_{1,0}[x'_{1,i_1}, v'_{1,0}] \}.$$

If  $w'_0 \in \gamma'_{1,0}[z'_{1,0}, v'_{1,0}]$ , by (3.8), we have

$$d_{D'_1}(w'_0) \geq \frac{1}{\rho_{18}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

If  $w'_0 \in \gamma'_{1,0}[z'_{1,0}, x'_{1,i_1}]$ , then Lemma 2.7 implies

$$k_{D_1}(z_{1,0}, w_0) \leq \int_{[z_{1,0}, w_0]} \frac{|dw|}{d_{D_1}(w)} \leq \frac{1}{\rho_2 - 1}$$

which together with Theorem G show that

$$\log \frac{d_{D'_1}(z'_{1,0})}{d_{D'_1}(w'_0)} \leq k_{D'_1}(z'_{1,0}, w'_0) < \mu_2$$

whence (3.8) yields

$$d_{D'_1}(w'_0) \geq \frac{1}{e^{\mu_2}} d_{D'_1}(z'_{1,0}) = \frac{1}{\rho_{18} e^{\mu_2}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

Hence

$$d_{D'_1}(w'_0) \geq \frac{1}{\rho_{18} e^{\mu_2}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]),$$

whence we know from (3.30) that Claim 3.9 holds.

**Claim 3.10.** Suppose  $w'_1, w'_2 \in \gamma'_{2,0}$ . If  $\ell(\gamma'_{2,0}[w'_1, y']) \leq 4\rho_{14}^{10}d_{D'_{1,0}}(y')$  for every  $y' \in \gamma'_{2,0}[w'_1, w'_2]$ , then

$$k_{D'_1}(w'_1, w'_2) \leq \frac{\log \rho_{18}}{\rho_{19}}$$

where  $\rho_{19} = [\rho_{17}^{\frac{223}{224}}]$ .

Suppose on the contrary that

$$k_{D'_1}(w'_1, w'_2) > \frac{\log \rho_{18}}{\rho_{19}}.$$

Then Theorem G shows

$$(3.31) \quad k_{D_1}(w_1, w_2) \geq \frac{1}{\mu_2} k_{D'_1}(w'_1, w'_2) \geq \frac{\log \rho_{18}}{\rho_{19}\mu_2}.$$

It follows from Lemma 2.7 that there exist  $s_1, s_2 \in \{1, \dots, k_1\}$  such that  $w_1 \in B_{1,s_1}$  and  $w_2 \in B_{1,s_2}$ . By Lemmas 3.2 and 3.3, we have

$$\begin{aligned} k_{D_1}(w_1, w_2) &\leq k_{D_1}(w_1, x_{1,s_1}) + k_{D_1}(x_{1,s_2}, x_{1,s_1}) + k_{D_1}(w_2, x_{1,s_2}) \\ &\leq \frac{2}{\rho_2 - 1} + 2\rho_6 \log \left( 1 + \frac{2\ell(\gamma_{1,0}[x_{1,s_2}, x_{1,s_1}])}{d_{D_1}(x_{1,s_2})} \right) \\ &\leq \frac{2}{\rho_2 - 1} + 2\rho_6 \log \left( 1 + 2\rho_6 \frac{d_{D_1}(x_{1,s_1})}{d_{D_1}(x_{1,s_2})} \right), \end{aligned}$$

which combining with (3.31) implies that

$$\frac{d_{D_1}(x_{1,s_1})}{d_{D_1}(x_{1,s_2})} \geq 5.$$

Hence Lemma 2.7 yields

$$\begin{aligned} k_{D_1}(w_1, w_2) &\leq \frac{2}{\rho_2 - 1} + 4c^2 \log \left( \frac{d_{D_1}(x_{1,s_1})}{d_{D_1}(x_{1,s_2})} \right) \\ &\leq \frac{2}{\rho_2 - 1} + 4c^2 \log \frac{\rho_2 + 1}{\rho_2 - 1} + 4c^2 \log \left( \frac{d_{D_1}(w_1)}{d_{D_1}(w_2)} \right), \end{aligned}$$

so that (3.31) yields

$$d_{D_1}(w_2) \leq \frac{1}{\frac{1}{8c^2\rho_{19}\mu_2}} d_{D_1}(w_1) \quad \text{and} \quad d_{D_1}(w_1) \leq 2|w_1 - w_2|.$$

Thus,

$$(3.32) \quad d_{D_1}(w_2) \leq \left( 2 / \rho_{18}^{\frac{1}{8c^2\rho_{19}\mu_2}} \right) |w_1 - w_2|.$$

By replacing the constant  $\mu_{10}$  with the one  $4\rho_{14}^{10}$ , it follows from Lemma 3.5 that there exists a  $\rho_7$ -uniform domain  $D'_{2,1} \subset D'_{1,0}$  and a point  $z'_{2,0} \in \gamma'_{2,0}[w'_1, w'_2]$  such that  $w'_1, w'_2 \in \overline{D'}_{2,1}$  with

$$(3.33) \quad \frac{1}{3 \cdot 2^{20} \rho_{14}^{20}} d_{D'_{1,0}}(z'_{2,0}) \leq d_{D'_{2,1}}(z'_{2,0}) \leq \frac{1}{2^{20} \rho_{14}^{20}} d_{D'_{1,0}}(z'_{2,0})$$

and

$$(3.34) \quad |z'_{2,0} - w'_2| \leq \frac{1}{2^{20}\rho_{14}^{20}} d_{D'_{1,0}}(z'_{2,0}).$$

Then (3.33) guarantees that

$$(3.35) \quad \begin{aligned} |z'_{2,0} - w'_1| &\leq \text{diam}(\gamma'_{2,0}[w'_1, z'_{2,0}]) \\ &\leq 4\rho_{14}^{10} d_{D'_{1,0}}(z'_{2,0}) \\ &\leq 3 \cdot 2^{22} \rho_{14}^{30} d_{D'_{2,1}}(z'_{2,0}). \end{aligned}$$

Since (3.34) assures that

$$k_{D'_{1,0}}(z'_{2,0}, w'_2) \leq \int_{[z'_{2,0}, w'_2]} \frac{|dw'|}{d_{D'_{1,0}}(w')} \leq \frac{1}{2^{20}\rho_{14}^{20} - 1},$$

we get from  $k_{D_{1,0}}(z_{2,0}, w_2) \leq \mu_2 (k_{D'_{1,0}}(z'_{2,0}, w'_2))^{\frac{1}{\mu_2}} \leq \log \frac{3}{2}$  that

$$(3.36) \quad |z_{2,0} - w_2| \leq \frac{1}{2} d_{D_{1,0}}(w_2),$$

which together with the inequality (3.32) yield

$$\begin{aligned} |z_{2,0} - w_1| &\geq |w_1 - w_2| - |z_{2,0} - w_2| \\ &\geq |w_1 - w_2| - \frac{1}{2} d_{D_{1,0}}(w_2) \\ &> \left(1 - \frac{1}{\rho_{18}^{\frac{1}{8c^2\rho_{19}\mu_2}}}\right) |w_1 - w_2|. \end{aligned}$$

Hence, Lemma 3.1 and the inequalities (3.32) and (3.36) show that

$$(3.37) \quad \begin{aligned} d_{D_{2,1}}(z_{2,0}) &\leq d_{D_{1,0}}(z_{2,0}) \\ &\leq \frac{3}{2} d_{D_{1,0}}(w_2) \\ &\leq \frac{3}{2(\rho_2 - 1)} d_{D_1}(w_2) \\ &\leq \frac{3}{(\rho_2 - 1) \left( \rho_{18}^{\frac{1}{8c^2\rho_{19}\mu_2 - 1}} - 1 \right)} |z_{2,0} - w_1|. \end{aligned}$$

Since for every  $z \in \mathbb{S}\left(z_{2,0}, \frac{1}{\rho_{11}} d_{D_{2,1}}(z_{2,0})\right)$ ,

$$\log \left(1 + \frac{1}{\rho_{11}}\right) \leq \log \left(1 + \frac{|z_{2,0} - z|}{d_{D_{2,1}}(z_{2,0})}\right) \leq k_{D_{2,1}}(z_{2,0}, z) \leq \int_{[z_{2,0}, z]} \frac{|dw|}{d_{D_{2,1}}(w)} \leq \frac{1}{\rho_{11} - 1},$$

we see from Theorem G that

$$k_{D'_{2,1}}(z'_{2,0}, z') \geq \left(\frac{1}{\mu_2} \log \left(1 + \frac{1}{\rho_{11}}\right)\right)^{\mu_2} \geq \left(\frac{2}{3\rho_{11}\mu_2}\right)^{\mu_2},$$

whence

$$f\left(\mathbb{S}(z_{2,0}, \frac{1}{\rho_{11}}d_{D_{2,1}}(z_{2,0}))\right) \subset \mathbb{R}^n \setminus \mathbb{B}\left(z'_{2,0}, \frac{1}{(3\rho_{11}\mu_2)^{\mu_2}}d_{D'_{2,1}}(z'_{2,0})\right).$$

Then for each  $x_{3,0} \in \mathbb{S}(z_{2,0}, \frac{1}{\rho_{11}}d_{D_{2,1}}(z_{2,0}))$ , by (3.35), we obtain

$$|z'_{2,0} - w'_1| \leq 3 \cdot 2^{22} \rho_{14}^{30} d_{D'_{2,1}}(z'_{2,0}) \leq \rho_{15} |x'_{3,0} - z'_{2,0}|.$$

We recall that  $\rho_{15} = 3^{1+\mu_2} \cdot 2^{22} \rho_{14}^{30} (\rho_{11}\mu_2)^{\mu_2}$ . By Lemma 3.6 and Theorem J, we see that  $f^{-1}$  is  $\eta_{K,\rho_5,\rho_8}$ -quasisymmetric. Hence

$$|z_{2,0} - w_1| \leq \eta_{K,\rho_5,\rho_8}(\rho_{15}) |x_{3,0} - z_{2,0}|.$$

But (3.37) and the choice of  $x_{3,0}$  imply

$$|z_{2,0} - w_1| \geq \frac{(\rho_2 - 1)\rho_{11} \left( \rho_{18}^{\frac{1}{8c^2\rho_{19}\mu_2}} - 1 \right)}{3} |x_{3,0} - z_{2,0}|.$$

This is the desired contradiction, from which we complete the proof of Claim 3.10.

**Claim 3.11.** *Suppose  $u'_1 \in \gamma'_{2,0}$ . If  $u'_2 \in \gamma'_{2,0}[u'_1, x'_{1,i_1}]$  and  $\ell(\gamma'_{2,0}[u'_1, u'_2]) \leq \rho_{13}d_{D'_{1,0}}(u'_2)$ , then for each  $u' \in \gamma'_{2,0}[u'_1, u'_2]$ , we have*

$$|u'_1 - u'| \leq \rho_{14}d_{D'_{1,0}}(u').$$

Suppose on the contrary that there exists an  $u' \in \gamma'_{2,0}[u'_1, u'_2]$  such that

$$(3.38) \quad |u'_1 - u'| > \rho_{14}d_{D'_{1,0}}(u').$$

By Lemma 2.7, we see that there exist  $p_i, q_i \in \{1, \dots, k_1\}$  such that

$$\gamma_{2,0}[u_1, u_2] \subset \cup_{j=p_i}^{q_i} B_{1,j}$$

and for  $j \in \{p_i, \dots, q_i\}$ ,  $\gamma_{2,0}[u_1, u_2] \cap B_{1,j} \neq \emptyset$ . Let

$$D_{1,i} = \cup_{j=p_i}^{q_i} B_{1,j}.$$

Then for each  $v_1, v_2 \in \overline{D}_{1,i}$ , there must exist  $o_1, o_2 \in \{p_i, \dots, q_i\}$  such that  $v_1 \in \overline{B}_{1,o_1}$  and  $v_2 \in \overline{B}_{1,o_2}$ . Let  $w_1 \in \gamma_{2,0}[u_1, u_2] \cap \overline{B}_{1,o_1}$  and  $w_2 \in \gamma_{2,0}[u_1, u_2] \cap \overline{B}_{1,o_2}$ . Then Claim 3.4 shows that

$$(3.39) \quad |u_2 - w_k| \leq \rho_9 d_{D_{1,0}}(u_2) \quad \text{for } k \in \{1, 2\}.$$

If  $|u_2 - w_k| \leq \frac{1}{2}d_{D_{1,0}}(w_k)$ , then we obviously have

$$d_{D_{1,0}}(u_2) \geq \frac{1}{2}d_{D_{1,0}}(w_k).$$

If  $|u_2 - w_k| > \frac{1}{2}d_{D_{1,0}}(w_k)$ , then

$$d_{D_{1,0}}(u_2) \geq \frac{1}{2\rho_9}d_{D_{1,0}}(w_k).$$

Thus, we have proved

$$(3.40) \quad d_{D_{1,0}}(u_2) \geq \frac{1}{2\rho_9}d_{D_{1,0}}(w_k).$$

It follows from  $d_{D_1}(w_1) \geq (1 - \frac{1}{\rho_2})d_{D_1}(x_{1,o_1})$ , Lemma 2.7 and Claim 3.5 that

$$d_{D_{1,0}}(x_{1,o_1}) \leq \frac{1}{\rho_2 - 1}d_{D_1}(w_1) \leq 2^{12}c\rho_3\rho_9\rho_{11}d_{D_{1,0}}(w_1).$$

Similarly, we obtain

$$d_{D_{1,0}}(x_{1,o_2}) \leq 2^{12}c\rho_3\rho_9\rho_{11}d_{D_{1,0}}(w_2).$$

Hence (3.39) and (3.40) show

$$\begin{aligned} |v_1 - v_2| &\leq |v_1 - w_1| + |w_2 - w_1| + |w_2 - v_2| \\ &\leq 2d_{D_{1,0}}(x_{1,o_1}) + 2d_{D_{1,0}}(x_{1,o_2}) + |w_1 - u_2| + |w_2 - u_2| \\ &\leq 2^{13}c\rho_3\rho_9\rho_{11}d_{D_{1,0}}(w_1) + 2^{13}c\rho_3\rho_9\rho_{11}d_{D_{1,0}}(w_2) + 2\rho_9d_{D_{1,0}}(u_2) \\ &\leq (2^{15}c\rho_3\rho_9^2\rho_{11} + 2\rho_9)d_{D_{1,0}}(u_2) \end{aligned}$$

which implies that

$$\delta_D(D_{1,i}) \leq (2^{15}c\rho_3\rho_9^2\rho_{11} + 2\rho_9)d_{D_{1,0}}(u_2) \leq \rho_{10}\rho_{11}d_{D_{1,0}}(u_2).$$

From Lemma 2.7, Claim 3.5 and the inequalities

$$\log \left( 1 + \frac{|v'_1 - w'_1|}{d_{D'_1}(w'_1)} \right) \leq k_{D'_1}(v'_1, w'_1) \leq \mu_2(k_{D_1}(v_1, w_1))^{\frac{1}{\mu_2}} \leq \mu_2,$$

we get

$$\begin{aligned} |v'_1 - w'_1| &\leq (e^{\mu_2} - 1)d_{D'_1}(w'_1) \\ &\leq (2^{15}c\rho_2\rho_3\rho_9\rho_{11}\mu_2)^{\mu_2}d_{D'_{1,0}}(w'_1) \\ &\leq (2^{15}c\rho_2\rho_3\rho_9\rho_{11}\mu_2)^{\mu_2}(d_{D'_{1,0}}(u'_2) + |w'_1 - u'_2|) \\ &\leq (2^{15}c\rho_2\rho_3\rho_9\rho_{11}\mu_2)^{\mu_2}(1 + \rho_{13})d_{D'_{1,0}}(u'_2). \end{aligned}$$

Similarly, we have

$$|v'_2 - w'_2| \leq (2^{15}c\rho_2\rho_3\rho_9\rho_{11}\mu_2)^{\mu_2}(1 + \rho_{13})d_{D'_{1,0}}(u'_2).$$

Then

$$\begin{aligned} |v'_1 - v'_2| &\leq |v'_1 - w'_1| + |w'_1 - w'_2| + |v'_2 - w'_2| \\ &\leq \rho_{13}^2d_{D'_{1,0}}(u'_2), \end{aligned}$$

so that

$$\delta_{D'}(D'_{1,i}) \leq \rho_{13}^2d_{D'_{1,0}}(u'_2).$$

It follows from Theorem K that  $D_{1,0}$  is a  $\varphi(\rho_5)$ -broad domain. From Theorem M, we also obtain that  $D'_{1,i}$  is  $\mu_5(a)$ -LLC<sub>2</sub> with respect to  $\delta_{D'}$ . Let  $w'_{0,i} \in \partial D'_{1,i}$  such that  $|w'_{0,i} - u'| = d_{D'_{1,0}}(u')$ . Then (3.38) and Claim 3.4 show that

$$\frac{|u' - u'_1|}{|u' - w'_{0,i}|} \geq \rho_{14} \frac{d_{D'_{1,0}}(u')}{d_{D'_{1,0}}(u')} \geq \rho_{14} \quad \text{and} \quad \frac{|u - u_1|}{|u - w_{0,i}|} \leq \rho_9.$$

But Theorem L shows that

$$\begin{aligned}\rho_{14} &\leq \frac{|u' - u'_1|}{|u' - w'_{0,i}|} \\ &\leq \mu_4(n, K, \mu_5(a), \varphi(\rho_5), \rho_{10}\rho_{11}, \rho_{13}^2) \frac{|u - u_1|}{|u - w_{0,i}|} \\ &\leq \rho_9\mu_4(n, K, \mu_5(a), \varphi(\rho_5), \rho_{10}\rho_{11}, \rho_{13}^2).\end{aligned}$$

This is a contradiction and so, we complete the proof of Claim 3.11.

In order to state the next claim, we let  $y'_0 \in \gamma'_{2,0}[x'_{1,i_1}, v'_{1,0}]$  be the first point along the direction from  $x'_{1,i_1}$  to  $v'_{1,0}$  such that

$$d_{D'_{1,0}}(y'_0) = \sup_{p' \in \gamma'_{2,0}[x'_{1,i_1}, v'_{1,0}]} d_{D'_{1,0}}(p').$$

It is possible that  $y'_0 = x'_{1,i_1}$  or  $v'_{1,0}$ . Clearly, there exists a nonnegative integer  $m$  such that

$$2^m d_{D'_{1,0}}(x'_{1,i_1}) \leq d_{D'_{1,0}}(y'_0) < 2^{m+1} d_{D'_{1,0}}(x'_{1,i_1}).$$

Let  $v'_0$  be the first point in  $\gamma'_{2,0}[x'_{1,i_1}, y'_0]$  from  $x'_{1,i_1}$  to  $y'_0$  satisfying

$$d_{D'_{1,0}}(v'_0) = 2^m d_{D'_{1,0}}(x'_{1,i_1}),$$

and let  $y'_1 = x'_{1,i_1}$ . If  $v'_0 = y'_1$ , we let  $y'_2 = y'_0$ . It is possible that  $y'_1 = y'_2$ . If  $v'_0 \neq y'_1$ , then we let  $y'_2, \dots, y'_{m+1} \in \gamma'_{2,0}[x'_{1,i_1}, y'_0]$  be the points such that for each  $i \in \{2, \dots, m+1\}$ ,  $y'_i$  denotes the first point from  $x'_{1,i_1}$  to  $y'_0$  with

$$d_{D'_{1,0}}(y'_i) = 2^{i-1} d_{D'_{1,0}}(y'_1).$$

Obviously,  $y'_{m+1} = y'_0$ . If  $v'_0 \neq y'_0$ , then we use  $y'_{m+2}$  to denote  $y'_0$ .

**Claim 3.12.**  $d_{D'_{1,0}}(y'_0) \geq \frac{1}{e^{\sqrt{\rho_{17}}}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$

Suppose on the contrary that

$$d_{D'_{1,0}}(y'_0) < \frac{1}{e^{\sqrt{\rho_{17}}}} \text{diam}(\gamma'_{1,0}[z'_1, z'_{1,0}]).$$

Then by Claim 3.1

$$k_{D'_{1,0}}(x'_{1,i_1}, v'_{1,0}) = \int_{\gamma'_{2,0}} \frac{|dw'|}{d(w')} \geq \frac{1}{4} e^{\sqrt{\rho_{17}}} > \rho_{10}^2 \rho_{11}^2 \rho_{17}$$

which is a contradiction to Claim 3.9, and so the proof of Claim 3.12 is complete.

For all  $i \in \{2, \dots, m+2\}$ , let  $y'_{0,i} \in \gamma'_{2,0}[y'_{i-1}, y'_i]$  such that

$$d_{D'_{1,0}}(y'_{0,i}) = \inf_{z' \in \gamma'_{2,0}[y'_{i-1}, y'_i]} d_{D'_{1,0}}(z').$$

**Claim 3.13.**  $d_{D'_{1,0}}(y'_i) \leq \rho_{14}^8 d_{D'_{1,0}}(y'_{0,i}).$

Suppose on the contrary that

$$(3.41) \quad d_{D'_{1,0}}(y'_i) > \rho_{14}^8 d_{D'_{1,0}}(y'_{0,i}).$$

Then there exists a nonnegative integer  $m_1$  such that

$$(3.42) \quad \frac{1}{2^{m_1+1}} d_{D'_{1,0}}(y'_{i-1}) \leq d_{D'_{1,0}}(y'_{0,i}) < \frac{1}{2^{m_1}} d_{D'_{1,0}}(y'_{i-1}).$$

Clearly,

$$(3.43) \quad 2^{m_1} \geq \frac{1}{4} \rho_{14}^8.$$

Let  $v'_{0,0}$  be the first point in  $\gamma'_{2,0}[y'_{i-1}, y'_{0,i}]$  from  $y'_{i-1}$  to  $y'_{0,i}$  with

$$d_{D'_{1,0}}(v'_{0,0}) = \frac{1}{2^{m_1}} d_{D'_{1,0}}(y'_{i-1}),$$

and let  $v'_{i,1} = y'_{i-1}$ . Further, we let  $v'_{i,2}, \dots, v'_{i,m_1+1} \in \gamma'_{2,0}[y'_{i-1}, y'_{0,i}]$  be points such that for each  $j \in \{2, \dots, m_1 + 1\}$ ,  $v'_{i,j}$  denotes the first point from  $y'_{i-1}$  to  $y'_{0,i}$  with

$$d(v'_{i,j}) = \frac{1}{2^{j-1}} d_{D'_{1,0}}(y'_{i-1}).$$

Obviously,  $v'_{i,m_1+1} = v'_{0,0}$ . We use  $v'_{i,m_1+2}$  to denote  $y'_{0,i}$ .

Next, we consider the remaining part, namely,  $\gamma'_{2,0}[y'_{0,i}, y'_i]$  of  $\gamma'_{2,0}[y'_{i-1}, y'_i]$ . Let  $u'_{0,0}$  be the first point from  $y'_{0,i}$  to  $y'_i$  satisfying

$$d_{D'_{1,0}}(u'_{0,0}) = 2^{m_1+1} d_{D'_{1,0}}(y'_{0,i}).$$

Let  $u'_{i,1} = y'_{0,i}$ . We take  $u'_{i,2}, \dots, u'_{i,m_1+2}$  such that for each  $j \in \{2, \dots, m_1 + 2\}$ ,  $u'_{i,j}$  denotes the first point from  $y'_{0,i}$  to  $y'_i$  with

$$(3.44) \quad d_{D'_{1,0}}(u'_{i,j}) = 2^{j-1} d_{D'_{1,0}}(u'_{i,1}),$$

and we use  $u'_{i,m_1+3}$  to denote  $y'_i$ .

**Proposition 3.4.**  $\ell(\gamma'_{2,0}[y'_{i-1}, y'_i]) \leq \rho_{13} d_{D'_{1,0}}(y'_i)$ .

The proof of the proposition follows by a method of contradiction. Suppose on the contrary that

$$\ell(\gamma'_{2,0}[y'_{i-1}, y'_i]) > \rho_{13} d_{D'_{1,0}}(y'_i).$$

Let  $w'_{1,i}$  be the last point in  $\gamma'_{2,0}[y'_{0,i}, y'_{i-1}]$  along the direction from  $y'_{0,i}$  to  $y'_{i-1}$  which satisfies

$$(3.45) \quad \ell(\gamma'_{2,0}[y'_{0,i}, w'_{1,i}]) \leq \rho_{12} d_{D'_{1,0}}(w'_{1,i}).$$

If  $w'_{1,i} \neq y'_{i-1}$ , then for each  $z' \in \gamma'_{2,0}[w'_{1,i}, y'_{i-1}]$ , we have

$$\ell(\gamma'_{2,0}[y'_{0,i}, w'_{1,i}]) + \ell(\gamma'_{2,0}[w'_{1,i}, z']) = \ell(\gamma'_{2,0}[y'_{0,i}, z']) > \rho_{12} d_{D'_{1,0}}(z')$$

which together with (3.45) imply

$$d_{D'_{1,0}}(z') < d_{D'_{1,0}}(w'_{1,i}) + \frac{1}{\rho_{12}} \ell(\gamma'_{2,0}[w'_{1,i}, z']).$$

Hence

$$(3.46) \quad \begin{aligned} \ell_{k_{D'_{1,0}}}(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]) &= \int_{\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]} \frac{|dw'|}{d_{D'_{1,0}}(w')} \\ &\geq \rho_{12} \log \left( 1 + \frac{\ell(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}])}{\rho_{12} d_{D'_{1,0}}(w'_{1,i})} \right). \end{aligned}$$

Next, we turn to get the following estimate:

$$(3.47) \quad d_{D'_{1,0}}(w'_{1,i}) \geq \frac{1}{2^{\frac{m_1}{2}}} \ell(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]).$$

Again we prove this inequality by a method of contradiction. Suppose on the contrary that

$$(3.48) \quad d_{D'_{1,0}}(w'_{1,i}) < \frac{1}{2^{\frac{m_1}{2}}} \ell(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]).$$

Then (3.43) and (3.46) show

$$(3.49) \quad k_{D'_{1,0}}(w'_{1,i}, y'_{i-1}) = \ell_{k_{D'_{1,0}}}(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]) \geq \rho_{12}.$$

Further, (3.43), (3.45) and (3.48) lead to

$$(3.50) \quad \ell(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]) \geq \frac{1}{2} \ell(\gamma'_{2,0}[y'_{0,i}, y'_{i-1}]).$$

It follows from (3.41) and (3.42) that

$$\ell(\gamma'_{2,0}[y'_{0,i}, y'_{i-1}]) \geq d_{D'_{1,0}}(y'_{i-1}) - d_{D'_{1,0}}(y'_{0,i}) \geq \frac{1}{2} d_{D'_{1,0}}(y'_{i-1}) \geq 2^{m_1-1} d_{D'_{1,0}}(y'_{0,i}).$$

It is obvious that there exists an integer  $n_1 \geq m_1 - 1$  such that

$$(3.51) \quad 2^{n_1} d_{D'_{1,0}}(y'_{0,i}) \leq \ell(\gamma'_{2,0}[y'_{0,i}, y'_{i-1}]) < 2^{n_1+1} d_{D'_{1,0}}(y'_{0,i})$$

and so, by Claims 3.6 and 3.7, Proposition 3.3 and (3.49), we have

$$(3.52) \quad \begin{aligned} \ell_{k_{D'_{1,0}}}(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]) &\leq 2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9 k_{D'}(w'_{1,i}, y'_{i-1}) \\ &\leq 2^{42} a^2 \rho_9^3 \rho_{10} \rho_{11}^3 \mu_2^8 \mu_9^3 \log \left( 1 + \frac{\ell(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}])}{d_{D'_{1,0}}(y'_{0,i})} \right) \\ &\leq \rho_{12}^{\frac{1}{2}} \log \left( 1 + \frac{\ell(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}])}{d_{D'_{1,0}}(y'_{0,i})} \right) \\ &\leq (n_1 + 1) \rho_{12}^{\frac{1}{2}}. \end{aligned}$$

If  $m_1 \geq \frac{n_1}{2}$ , then by (3.43), (3.46) and (3.48),

$$(3.53) \quad \ell_{k_{D'_{1,0}}}(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]) \geq \rho_{12} \log \left( 1 + \frac{2^{\frac{m_1}{2}}}{\rho_{12}} \right) > \frac{1}{24} m_1 \rho_{12} > \frac{1}{48} n_1 \rho_{12}.$$

For the case  $m_1 < \frac{n_1}{2}$ , we get from (3.42), (3.43), (3.46), (3.50) and (3.51) that

$$(3.54) \quad \ell_{k_{D'_{1,0}}}(\gamma'_{2,0}[w'_{1,i}, y'_{i-1}]) \geq \rho_{12} \log \left( 1 + \frac{2^{\frac{n_1}{2}}}{8 \rho_{12}} \right) > \frac{1}{24} n_1 \rho_{12}.$$



Obviously, the inequalities (3.53) and (3.54) contradict the inequality (3.52), which shows that the inequality (3.47) holds.

If  $|w'_{1,i} - y'_{i-1}| \leq \frac{1}{2}d_{D'_{1,0}}(y'_{i-1})$ , then  $d_{D'_{1,0}}(w'_{1,i}) \geq \frac{1}{2}d_{D'_{1,0}}(y'_{i-1})$ . On the other hand, if  $|w'_{1,i} - y'_{i-1}| > \frac{1}{2}d_{D'_{1,0}}(y'_{i-1})$ , then it follows from (3.47) that

$$d_{D'_{1,0}}(w'_{1,i}) \geq \frac{1}{2^{\frac{m_1}{2}+1}}d_{D'_{1,0}}(y'_{i-1}).$$

Hence, we obtain from (3.42) that

$$(3.55) \quad d_{D'_{1,0}}(w'_{1,i}) \geq 2^{\frac{m_1}{2}-1}d_{D'_{1,0}}(y'_{0,i}).$$

We claim

$$(3.56) \quad \ell(\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}]) \leq \rho_{12} d_{D'_{1,0}}(w'_{1,i}).$$

Otherwise, (3.45) leads to

$$(3.57) \quad \ell(\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}]) > \rho_{12} d_{D'_{1,0}}(w'_{1,i}) > \ell(\gamma'_{2,0}[y'_{0,i}, w'_{1,i}]).$$

Also (3.55) yields

$$\ell(\gamma'_{2,0}[y'_{0,i}, w'_{1,i}]) \geq d_{D'_{1,0}}(w'_{1,i}) - d_{D'_{1,0}}(y'_{0,i}) \geq \frac{1}{2}d_{D'_{1,0}}(w'_{1,i}).$$

Then we deduce from (3.55) and (3.57) that

$$(3.58) \quad \ell(\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}]) > 2^{\frac{m_1}{2}-2}d_{D'_{1,0}}(y'_{0,i}).$$

For every  $w' \in \gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}]$ , by (3.44), we have

$$d_{D'_{1,0}}(w') \leq d_{D'_{1,0}}(u'_{i, [\frac{m_1}{4}]}) \leq 2^{\frac{m_1}{4}}d_{D'_{1,0}}(y'_{0,i}).$$

Hence (3.43) and (3.58) imply

$$(3.59) \quad \begin{aligned} k_{D'_{1,0}}(u'_{i, [\frac{m_1}{4}]}, y'_{0,i}) &= \int_{\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}]} \frac{|dw'|}{d_{D'_{1,0}}(w')} \\ &\geq \frac{\ell(\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}])}{2^{\frac{m_1}{4}}d_{D'_{1,0}}(y'_{0,i})} \\ &\geq \rho_{12}, \end{aligned}$$

and so, from Proposition 3.3, Claims 3.6 and 3.7, we obtain that

$$\begin{aligned} k_{D'_{1,0}}(u'_{i, [\frac{m_1}{4}]}, y'_{0,i}) &\leq 2^{12}\rho_9\rho_{11}\mu_2^2\mu_9 k_{D'}(u'_{i, [\frac{m_1}{4}]}, y'_{0,i}) \\ &\leq 2^{42}a^2\rho_9^3\rho_{10}\rho_{11}^3\mu_2^8\mu_9^3 \log \left( 1 + \frac{\text{diam}(\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}])}{d_{D'_{1,0}}(y'_{0,i})} \right). \end{aligned}$$

This contradicts the inequality (3.59). Thus, (3.56) holds.

Now, we are in a position to prove Proposition 3.4. By (3.45) and (3.56), we have

$$\ell(\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, w'_{1,i}]) = \ell(\gamma'_{2,0}[u'_{i, [\frac{m_1}{4}]}, y'_{0,i}]) + \ell(\gamma'_{2,0}[y'_{0,i}, w'_{1,i}]) \leq 2\rho_{12} d_{D'_{1,0}}(w'_{1,i}).$$

Thus, on one hand, Claim 3.11 and (3.43) yield

$$\left| u'_{i, [\frac{m_1}{4}]} - y'_{0,i} \right| \leq \rho_{14} d_{D'_{1,0}}(y'_{0,i}) \leq 2^{\frac{m_1}{4}-3} d_{D'_{1,0}}(y'_{0,i}).$$

But on the other hand, the inequality (3.44) shows that

$$\left| u'_{i, [\frac{m_1}{4}]} - y'_{0,i} \right| \geq d_{D'_{1,0}}(u'_{i, [\frac{m_1}{4}]}) - d_{D'_{1,0}}(y'_{0,i}) \geq \frac{1}{2} d_{D'_{1,0}}(u'_{i, [\frac{m_1}{4}]}) \geq 2^{[\frac{m_1}{4}]-2} d_{D'_{1,0}}(y'_{0,i}).$$

This is a contradiction and so, the proof of Proposition 3.4 is complete.

Now the proof of Claim 3.13 is easy. Indeed, by (3.41), on one hand we have

$$(3.60) \quad |y'_i - y'_{0,i}| \geq d_{D'_{1,0}}(y'_i) - d_{D'_{1,0}}(y'_{0,i}) \geq (\rho_{14}^8 - 1) d_{D'_{1,0}}(y'_{0,i}),$$

and on the other hand, Claim 3.11 and Proposition 3.4 show that

$$|y'_i - y'_{0,i}| \leq \rho_{14} d_{D'_{1,0}}(y'_{0,i}),$$

which contradicts (3.60). Thus, Claim 3.13 is proved.

**Claim 3.14.**  $\ell(\gamma'_{2,0}[y'_{i-1}, y'_i]) \leq \rho_{14}^2 d_{D'_{1,0}}(y'_i)$ .

Suppose on the contrary that

$$\ell(\gamma'_{2,0}[y'_{i-1}, y'_i]) > \rho_{14}^2 d_{D'_{1,0}}(y'_i).$$

Then

$$k_{D'_{1,0}}(y'_{i-1}, y'_i) = \int_{\gamma'_{2,0}[y'_{i-1}, y'_i]} \frac{|dw'|}{d_{D'_{1,0}}(w')} \geq \frac{\ell(\gamma'_{2,0}[y'_{i-1}, y'_i])}{d_{D'_{1,0}}(y'_i)} > \rho_{14}^2,$$

and so, Proposition 3.3, Claims 3.6 and 3.7 show that

$$\begin{aligned} k_{D'_{1,0}}(y'_{i-1}, y'_i) &\leq 2^{12} \rho_9 \rho_{11} \mu_2^2 \mu_9 k_{D'}(y'_{i-1}, y'_i) \\ &\leq 2^{42} a^2 \rho_9^3 \rho_{10}^3 \rho_{11}^3 \mu_2^8 \mu_9^3 \log \left( 1 + \frac{\ell(\gamma'_{2,0}[y'_i, y'_{i-1}])}{d_{D'_{1,0}}(y'_{0,i})} \right). \end{aligned}$$

Thus, Claim 3.13 yields that

$$\begin{aligned} \frac{\ell(\gamma'_{2,0}[y'_{i-1}, y'_i])}{\rho_{14}^8 d_{D'_{1,0}}(y'_{0,i})} &\leq \frac{\ell(\gamma'_{2,0}[y'_{i-1}, y'_i])}{d_{D'_{1,0}}(y'_i)} \\ &\leq k_{D'_{1,0}}(y'_{i-1}, y'_i) \\ &\leq 2^{42} a^2 \rho_9^3 \rho_{10}^3 \rho_{11}^3 \mu_2^8 \mu_9^3 \log \left( 1 + \frac{\ell(\gamma'_{2,0}[y'_i, y'_{i-1}])}{d_{D'_{1,0}}(y'_{0,i})} \right) \\ &\leq \rho_{13} \log \left( 1 + \frac{\ell(\gamma'_{2,0}[y'_i, y'_{i-1}])}{d_{D'_{1,0}}(y'_{0,i})} \right) \end{aligned}$$

which is a contradiction. The proof of Claim 3.14 is complete.

**Claim 3.15.** For  $z' \in \gamma'_{2,0}[y'_1, y'_0]$ , we have  $\ell(\gamma'_{2,0}[y'_1, z']) \leq 4\rho_{14}^{10} d_{D'_{1,0}}(z')$ .

We divide the proof into two cases. In case there exists  $k \in \{1, \dots, m\}$  such that  $z' \in \gamma'_{2,0}[y'_k, y'_{k+1}]$ . Then for  $k = 1$ , the result easily follows from Claims 3.13 and 3.14. For  $k > 1$ , we deduce from Claims 3.13 and 3.14 that

$$\begin{aligned} \ell(\gamma'_{2,0}[y'_1, z']) &= \ell(\gamma'_{2,0}[y'_1, y'_2]) + \dots + \ell(\gamma'_{2,0}[y'_{k-1}, y'_k]) + \ell(\gamma'_{2,0}[y'_k, z']) \\ &\leq 2\rho_{14}^2(d_{D'_{1,0}}(y'_1) + \dots + d_{D'_{1,0}}(y'_{k-1}) + d_{D'_{1,0}}(y'_k)) \\ &\leq 2\rho_{14}^{10}d_{D'_{1,0}}(z'). \end{aligned}$$

In the remaining case where  $z' \in \gamma'_{2,0}[y'_{m+1}, y'_0]$ , Claims 3.13 and 3.14 imply that

$$\begin{aligned} \ell(\gamma'_{2,0}[y'_1, z']) &\leq 2\rho_{14}^2(d_{D'_{1,0}}(y'_1) + \dots + d_{D'_{1,0}}(y'_m) + d_{D'_{1,0}}(y'_{m+1})) \\ &\leq 4\rho_{14}^2d_{D'_{1,0}}(y'_{m+1}) \\ &\leq 4\rho_{14}^{10}d_{D'_{1,0}}(z') \end{aligned}$$

and so, Claim 3.15 holds.

Now we are ready to establish the proof of Lemma 3.8.

It follows from Claims 3.12 and (3.10) that

$$(3.61) \quad k_{D'_{1,0}}(y'_1, y'_0) \geq \log \frac{d_{D'_{1,0}}(y'_0)}{d_{D'_{1,0}}(y'_1)} \geq \frac{1}{2}\rho_{17}.$$

Then Claim 3.6 yields

$$(3.62) \quad k_{D'_1}(y'_1, y'_0) \geq \frac{1}{2^{12}\rho_9\rho_{11}\mu_2^2\mu_9} k_{D'_{1,0}}(y'_1, y'_0) \geq \rho_{19}.$$

Claim 3.15 shows that (3.62) contradicts Claim 3.10. Thus the proof of Lemma 3.8 is complete.  $\square$

Finally, the following result follows from the similar reasoning as in the proof of Lemma 3.8.

**Lemma 3.9.** *For each  $z' \in \beta'_{1,0}$ ,  $\text{diam}(\beta'_{1,0}[z'_2, z']) < \rho_{18}d_{D'_1}(z')$ .*

**The proof of Theorem 1.1.** Let  $x' \in \gamma'_{1,0}[z'_1, z'_0] \cap \beta'_{1,0}[z'_2, z'_0]$  such that

$$\gamma'_{1,0}[z'_1, x'] \cap \beta'_{1,0}[z'_2, x'] = \{x'\},$$

and let  $\lambda' = \gamma'_{1,0}[z'_1, x'] \cup \beta'_{1,0}[z'_2, x']$ . Then Lemmas 3.8 and 3.9 show that for every  $z' \in \lambda'$ ,

$$\min_{i=1,2} \{\text{diam}(\lambda'[z'_i, z'])\} \leq \rho_{18}d_{D'_1}(z').$$

It follows from Theorem E that Theorem 1.1 holds.  $\square$

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